INTRODUCTION TO

TROPICAL CURVE COUNTING

Tropical Geometry School

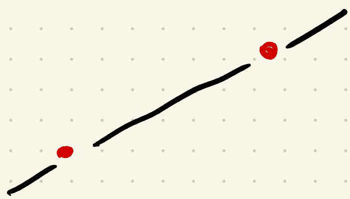
Valladolid, 5/23-27/2022

- Renzo Cavalieri -

Introduction

Enumerative geometry:

Q_1: how many lines in the plane pass through 2 points?



A = 1

Q_d: how many rat'l curves of deg d pass through $3d-1$ points?

A_2: 1 conic through 5 pts

A_3: 12 cubics — 8 pts

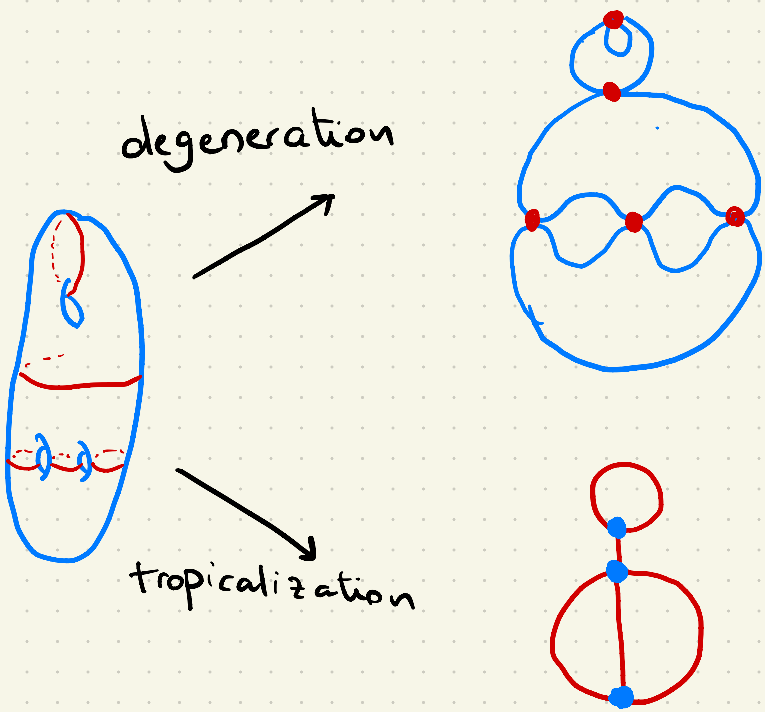
A_4: 620 quartics

$A_{5,\dots}$ very complicated

Tropical geometry:

(1) Generalization of toric geometry.

(2) Degenerations of curves.



Kontsevich / Mikhalkin:

Answers Q_d for every d.

(1) Think of curves as images of maps.

(2) Construct a moduli space for all possible maps.

(3) Interpret "passing through a point" as a subspace.

(4) Passing through multiple points = intersecting.

(5) Recursion.

Plan for the course:

1. Basics about cones, fans and their maps.

2. Abstract tropical curves & moduli spaces.

3. Tropical stable maps.

4. Study the recursion.

   - Rec counting N_d^{trop}
   - Corresp theorem

$$N_d^{trop} = N_d$$

Lecture 1

Cones & Fans

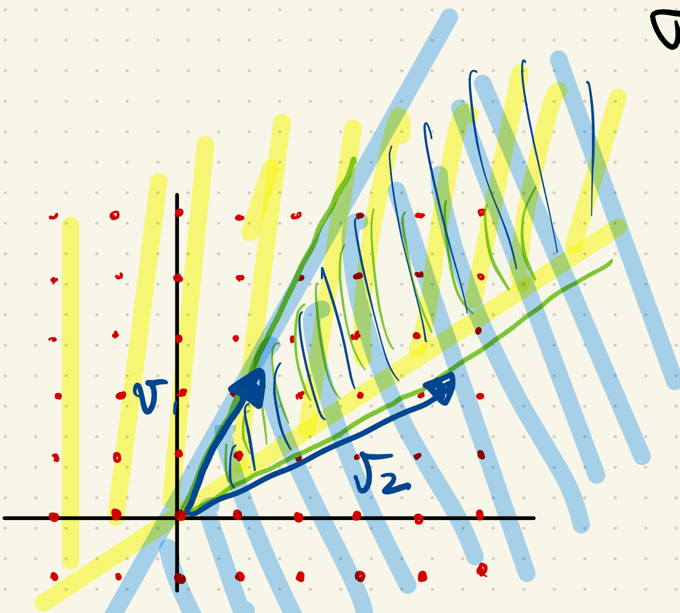
Rational polyhedral cone:

- Intersection of half spaces

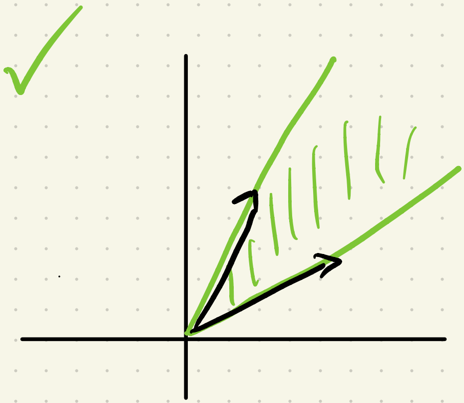
$$\sigma = \bigcap H^+$$

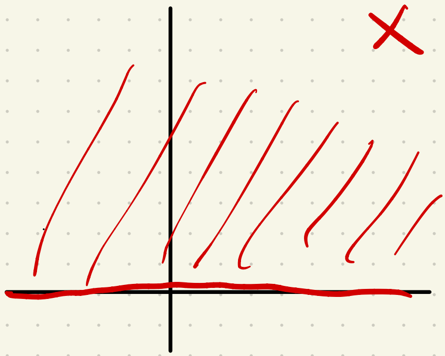
- Non-negative span of vectors

$$\sigma = \left\{ \lambda_i v_i \,\middle|\, \lambda_i \in \mathbb{R}_{\geq 0} \right\}$$

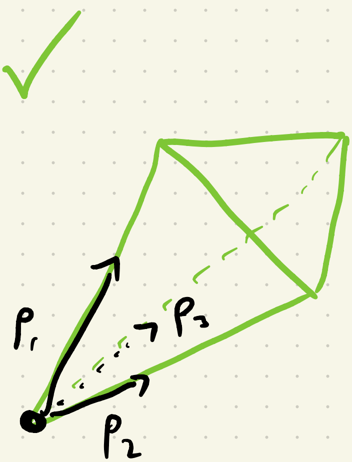


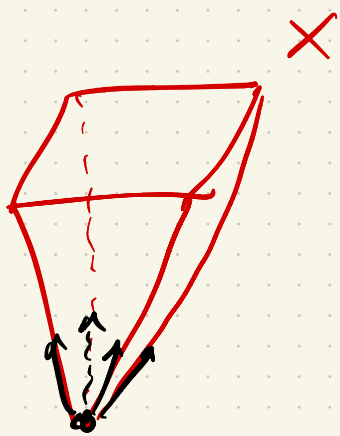
Strongly convex: if v and $-v \in \sigma$
$\Rightarrow v = 0$





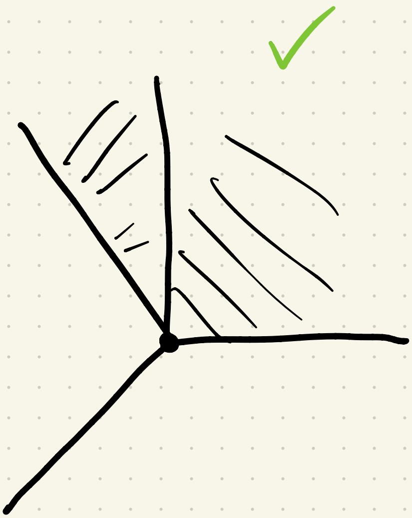
Simplicial: strongly cvx +
same # of rays as dim.



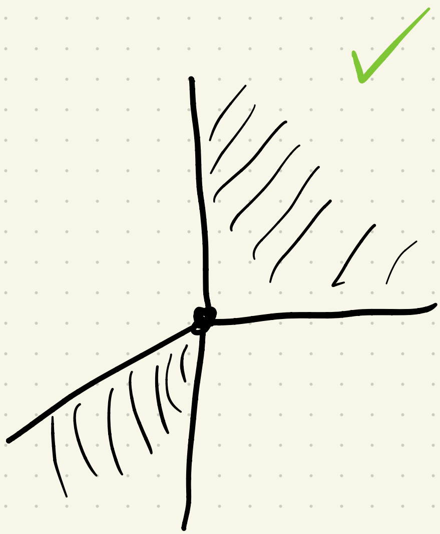


Rational polyhedral fan:

collection of R.P. cones meeting along faces



NOT PURE DIM



PURE DIM 2

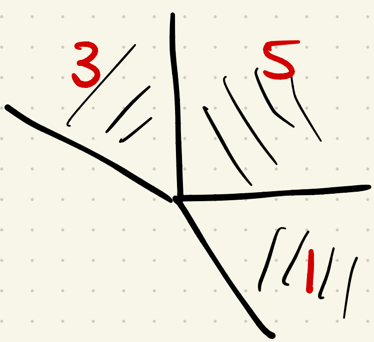
Maximal cone: cones that are not faces of any other cone

Pure dimensional fan: all max cones have same dimension.

Weight function:

Given a R.P. fan Σ a weight function is

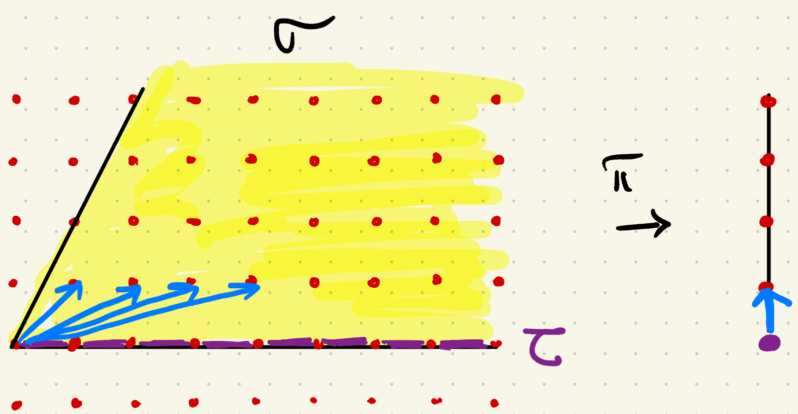
$$\omega : \left\{ \begin{matrix} \text{maximal} \\ \text{cones of} \\ \Sigma \end{matrix} \right\} \longrightarrow \mathbb{Z}_{\geq 0}$$



Normal vector $u_{\tau/\sigma}$:

any integral vector in σ that descends to a generator of the lattice of $\dfrac{\mathrm{Span}(\sigma)}{\mathrm{Span}(\tau)}$

Image of $\mathbb{Z}^n \cap \mathrm{Span}(\sigma)$



Balanced fan: given a cod 1 cone τ, we say Σ is balanced at τ if we have

$$\sum_{\sigma > \tau} \omega(\sigma) \cdot u_{\tau/\sigma} \in \mathrm{Span}(\tau)$$

Marking a choice of a integral vector (not necessarily primitive) for each ray of Σ

Marking => weight function

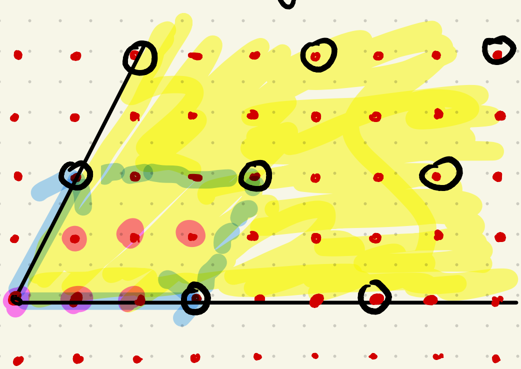
lattice index

$$\omega_M(\sigma) = \left| \frac{\operatorname{Span}(\sigma) \cap \mathbb{Z}^n}{\mathbb{Z}v_1 \oplus \dots \oplus \mathbb{Z}v_n} \right|$$

$$= \left| \det \left[v_1 \middle| \dots \middle| v_n \right] \right|$$

write v_i's in terms of generators of lattice of $\mathbb{Z}^n \cap \operatorname{Span}(\sigma)$

σ



$v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$v_2 = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$

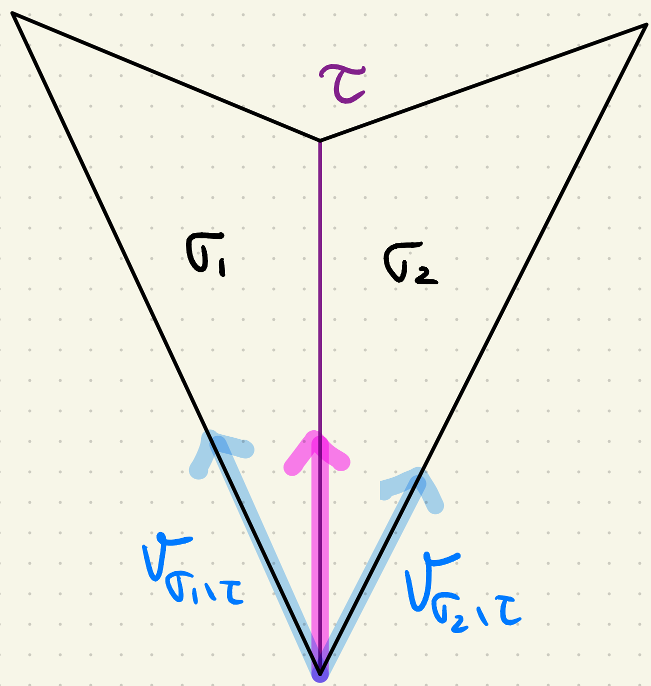
$$\left| \det \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix} \right| = 6$$

Lemma Σ a marked fan

τ codim 1 face of Σ

$\sigma > \tau$

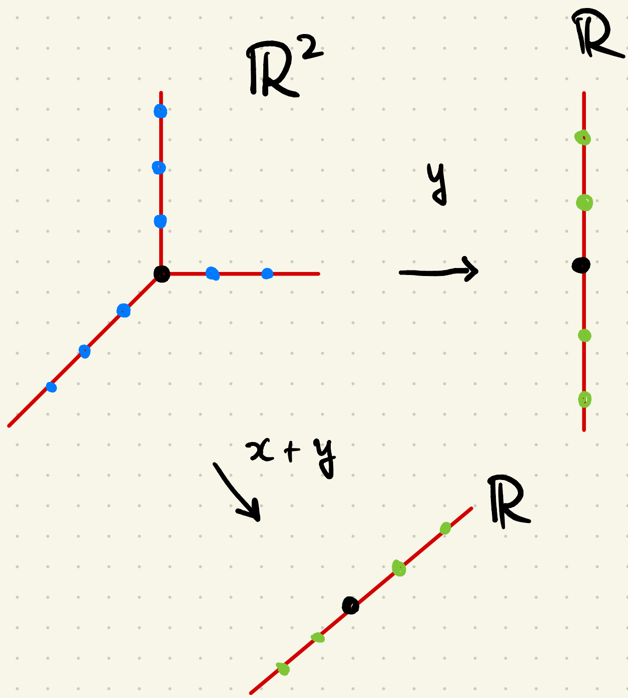
$$\boxed{\Sigma \text{ balanced} \iff \sum_{\sigma > \tau} v_{\sigma \setminus \tau} \in \langle \tau \rangle}$$



Map of fans: $\Sigma_1 \subseteq \mathbb{R}^{n_1}$, $\Sigma_2 \subseteq \mathbb{R}^{n_2}$

$$f : |\Sigma_1| \longrightarrow |\Sigma_2|$$

is a map of fans if it is the restriction of an integral linear function $L : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$

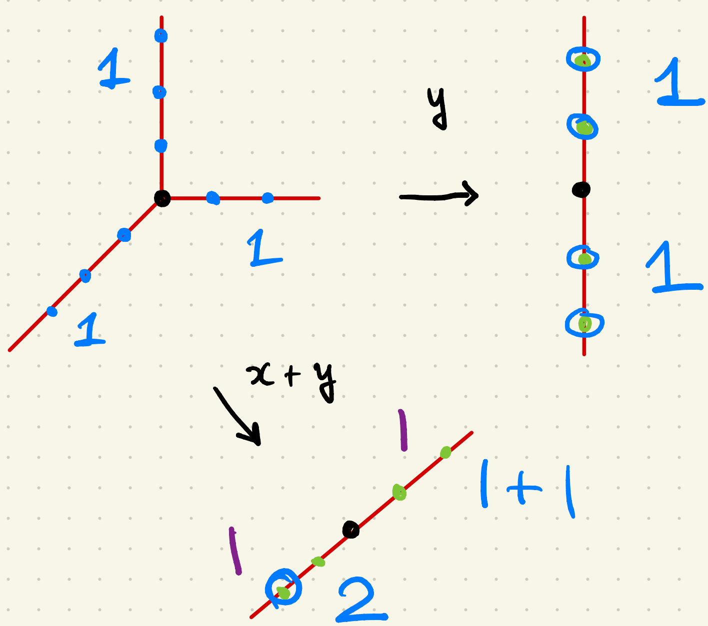


Push-forward of a fan: $f: \Sigma_1 \to \Sigma_2$, $\quad \Sigma_1 \subseteq \mathbb{R}^{n_1}$, $\Sigma_2 \subseteq \mathbb{R}^{n_2}$

$$f_*(\Sigma_1) = \{ f(\sigma_1) \mid f \text{ is injective on } \sigma_1 \}$$

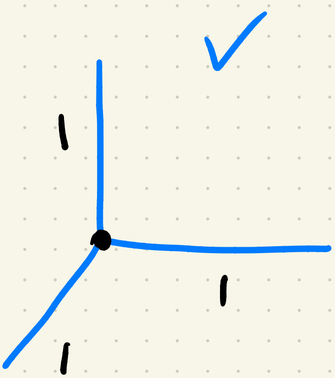
If Σ_1 has a weight function $\Rightarrow$ want to give a weight function to $f_*(\Sigma_1)$

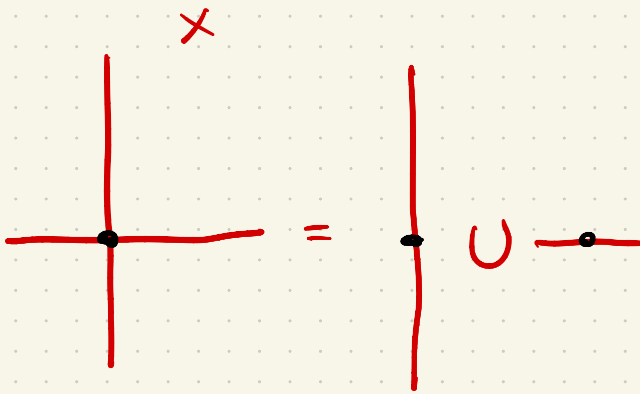
$$\omega_{f_*(\Sigma_1)}(f(\sigma)) = \omega_{\Sigma_1}(\sigma) \cdot \left| \frac{f(\sigma) \cap \mathbb{Z}^{n_2}}{f(\sigma \cap \mathbb{Z}^{n_1})} \right|$$



Irreducible fan: Σ a balanced fan if it cannot be decomposed as a $\cup$ of balanced subfans (w non-identical support)

⚠ For a red fan, decomposition into irred components is not necessarily unique.





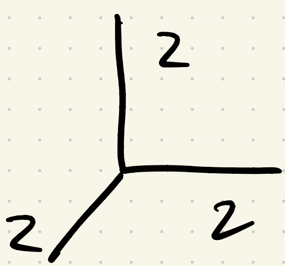
Lemma: Σ_1, Σ_2 balanced fans of dimension d.

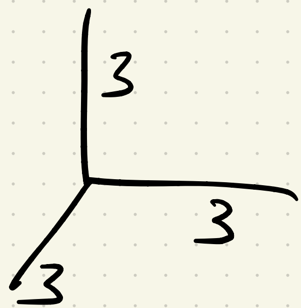
- Σ_1 irreducible
- $|\Sigma_2| \subseteq |\Sigma_1|$

$\Rightarrow \quad \exists \ \lambda \in \mathbb{Q}$ s.t.

$$\boxed{\Sigma_1 = \lambda \Sigma_2}$$

e.g.





Degree of $f: \Sigma_1 \to \Sigma_2$

two balanced fans, Σ_2 irreducible.

$f(\sigma_1) = \sigma_2$

- $$\mathrm{mult}_f(\sigma_1) = \frac{\omega_{\Sigma_1}(\sigma_1)}{\omega_{\Sigma_2}(\sigma_2)} \cdot \left| \frac{\sigma_2 \cap \mathbb{Z}^{n_2}}{f(\sigma_1 \cap \mathbb{Z}^{n_1})} \right|$$

- $$\deg f := \sum_{\substack{\sigma \text{ s.t.} \\ f(\sigma) = \sigma_2}} \mathrm{mult}_f(\sigma)$$

Σ_2 IRREDUCIBLE $\Rightarrow$ deg is well defined, i.e. choose σ_2 however you want.

VERY IMPORTANT TRICK!

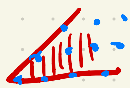
If Σ_1 and Σ_2 are marked fans (with markings $\{v_i\}$, $\{w_i\}$) and are given weight function determined by markings, then the local degree of f at a cone σ is given by

$$\left| \det M_{f|\mathrm{span}(\sigma)} \right|$$

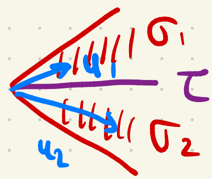
given bases $\{v_i\}$ (in σ)
$\{w_j\}$ (in $f(\sigma)$)

Recap:

(1) Rational polyhedral cones & fans



(2) Weight functions + balancing



$$\sum w_i u_{\tau/\sigma_i} \in \langle \tau \rangle$$

(3) Markings
- ↳ weights
- ↳ easy way to check balancing

(4) Maps of fans $\quad f: \Sigma_1 \to \Sigma_2$
- ↳ local degree
- ↳ if Σ_2 irred $\Rightarrow \deg(f)$ well defined

(5) Marking $\Rightarrow$ easy way to compute local degrees.

Lecture 2

Abstract tropical curves & their moduli

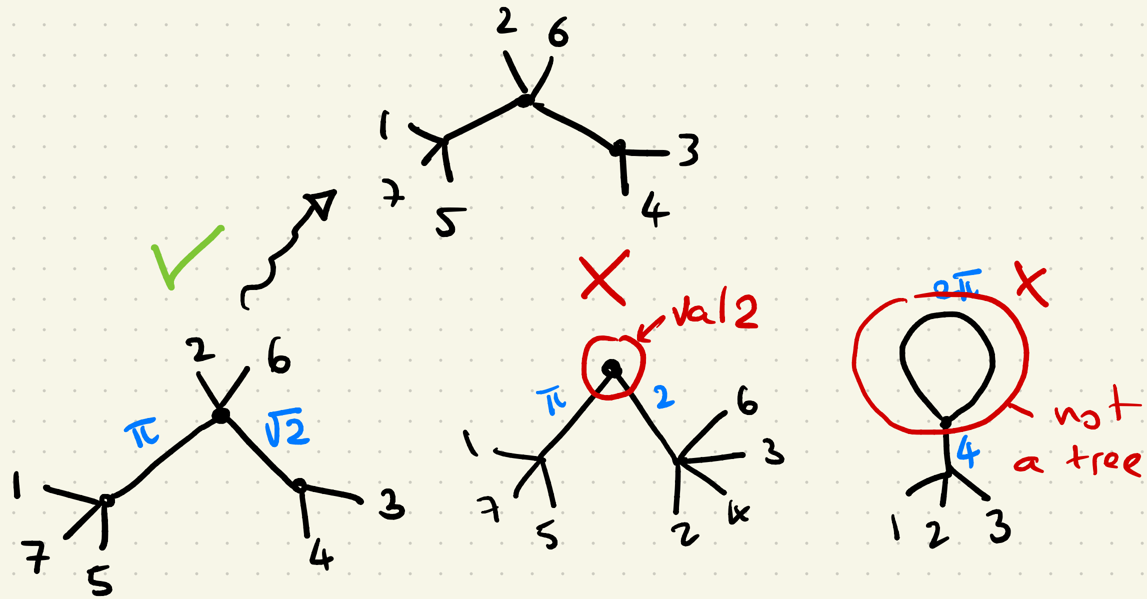
Goals:

- Introduce $M_{0,n}^{trop}$
- Realize it as a balanced fan

Abstract, rational, n-pointed, stable tropical curve:

- Metric graph ($m: E(\Gamma) \to \mathbb{R}_{\geq 0}$)
- Trees
- n labeled leaves
- every vertex is at least 3-valent

Topological type: forget the metric

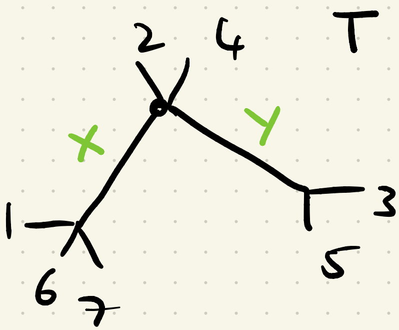


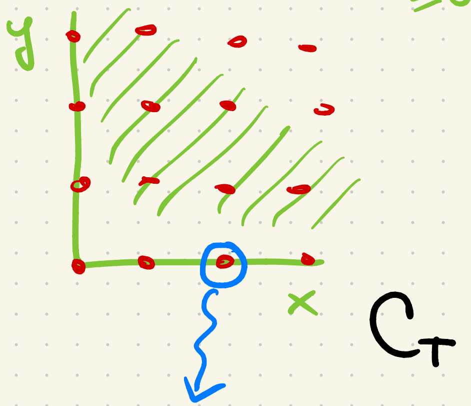
$\mathcal{M}_{0,n}^{trop}$: parameter space for all a. r. s. n. ptd tropical curves.

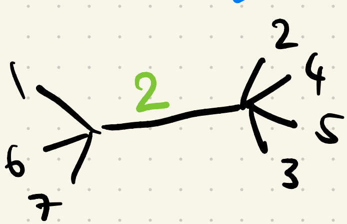
rational i.e. genus of t. curve

Curves of a given topological type are parametrize by a cone $\cong \mathbb{R}_{\geq 0}^{|E(\Gamma)|}$

$\mathbb{N}^2 \subseteq \mathbb{R}^2_{\geq 0}$

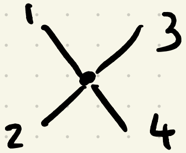




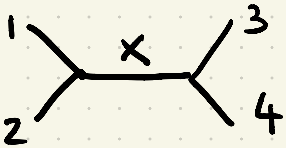


$$\mathcal{M}_{0,n}^{trop} = \coprod_{\text{top types } T} C_T \Big/ \sim$$

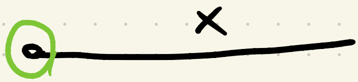
$\mathcal{M}_{0,4}^{trop}$:

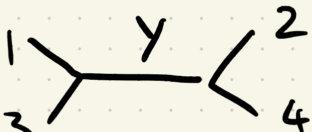


T_0 $\rightsquigarrow$ C_{T_0} =

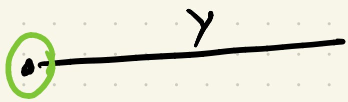


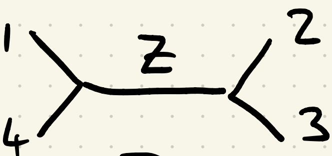
T_1 $\rightsquigarrow$ C_{T_1} =



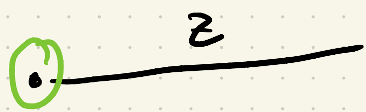


T_2 $\rightsquigarrow$ C_{T_2} =

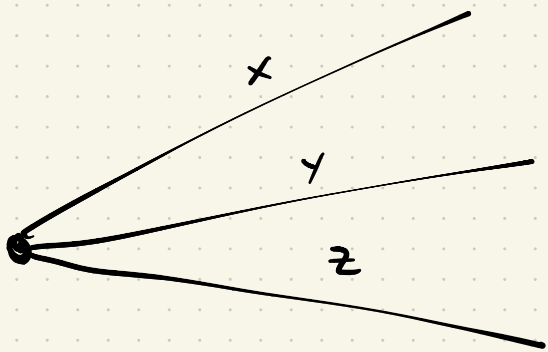




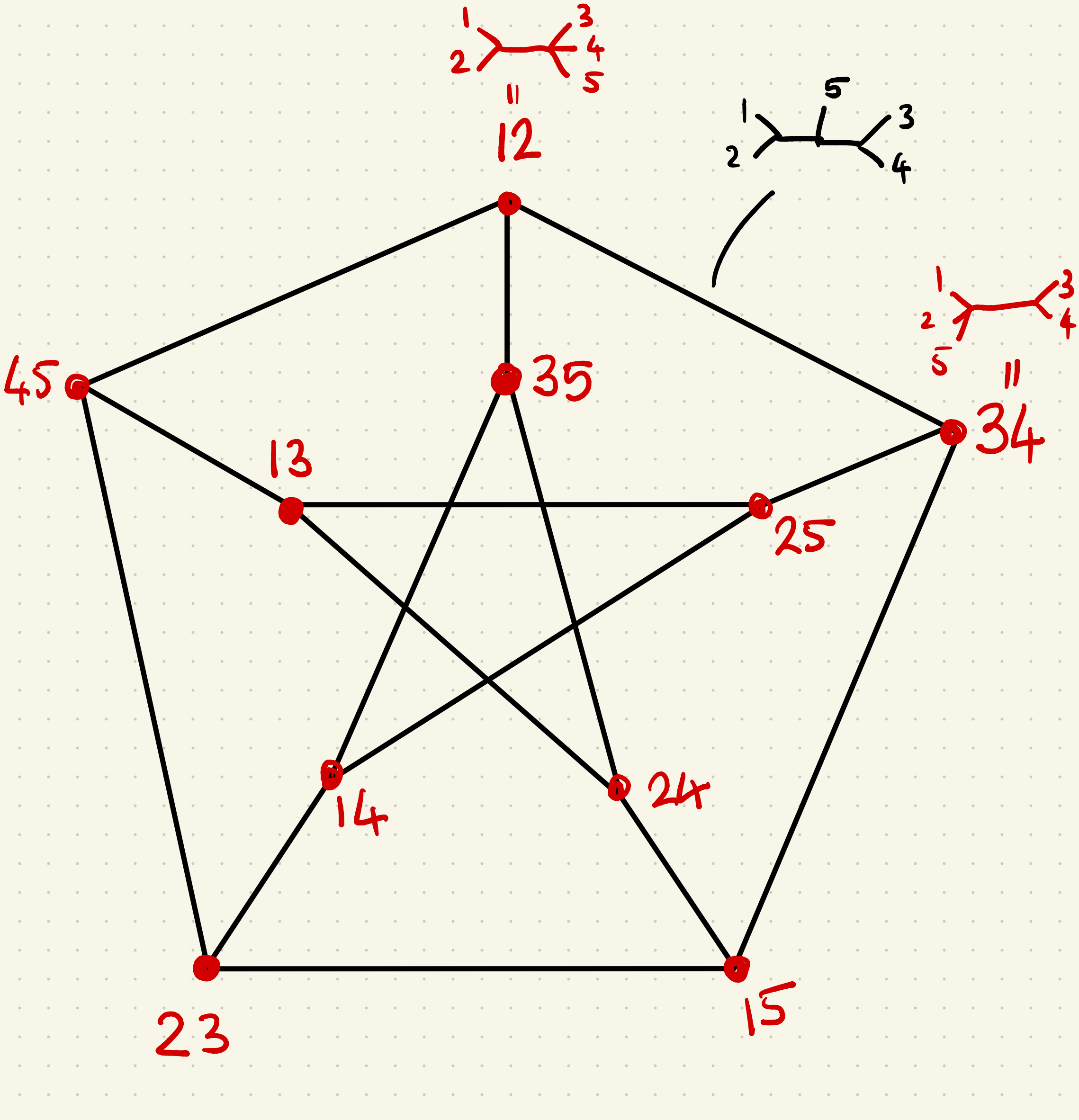
T_3 $\rightsquigarrow$ C_{T_3} =



$M_{0,4}^{trop}$ =



$\mathcal{M}_{0,5}^{trop}$

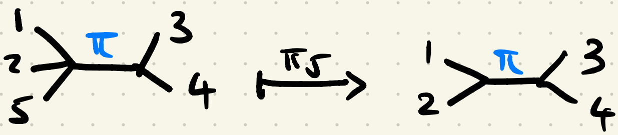


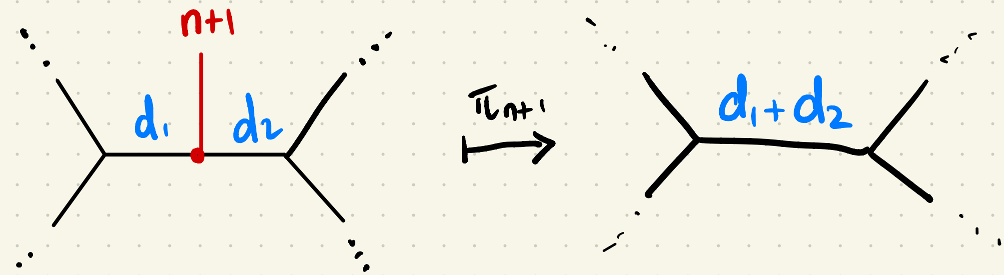
Forgetful morphisms:

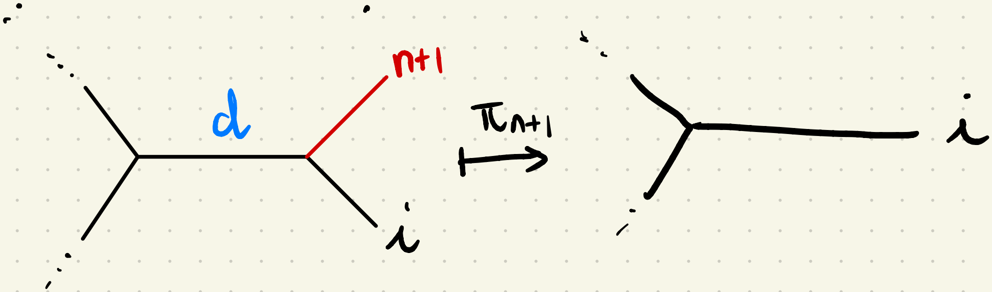
$$\pi_{n+1} : \mathcal{M}^{trop}_{0,n+1} \longrightarrow \mathcal{M}^{trop}_{0,n}$$

$\Gamma \longmapsto$ Remove the end labeled $n+1$ from Γ + stab. if needed

E.g.





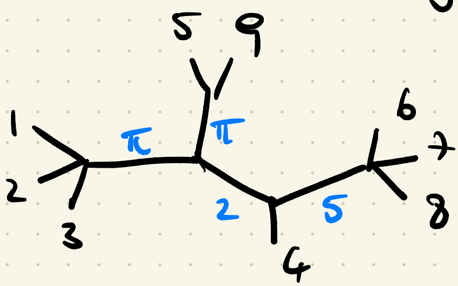


The distance function:

$$\mathrm{dist}_n : \mathcal{M}_{0,n}^{trop} \longrightarrow \mathbb{R}^{\binom{n}{2}}$$

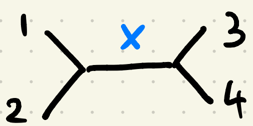
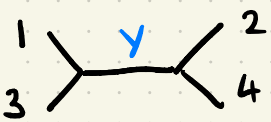
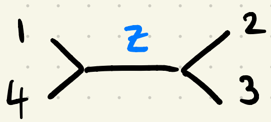
$$\Gamma \longmapsto \begin{pmatrix} \text{length of} \\ \text{path from} \\ \text{end } i \text{ to} \\ \text{end } j \end{pmatrix}_{ij}$$

Example



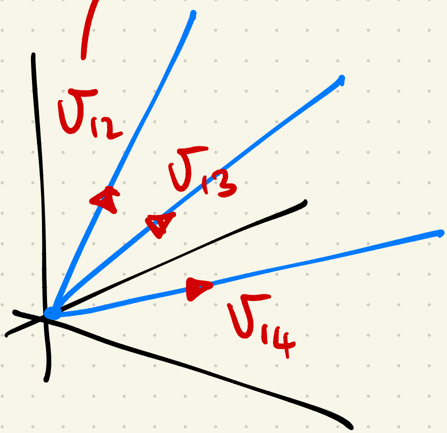
$\mathrm{dist}(\Gamma)_{15} = 2\pi$

$\mathrm{dist}(\Gamma)_{58} = \pi + 7$

		12	13	14	23	24	34
	$\mapsto$	(0	x	x	x	x	0)
	$\mapsto$	(y	0	y	y	0	y)
	$\mapsto$	(z	z	0	0	z	z)

$\mathrm{dist}\left(M_{0,4}^{trop}\right) \subseteq \mathbb{R}^6$

$:= \mathrm{dist}\left({}^{1}_{2}\!>\!\overset{1}{\text{———}}\!<^{3}_{4} \right)$

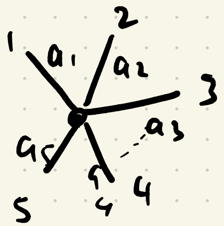


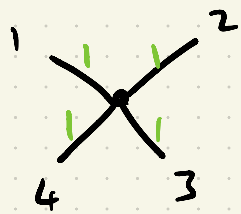
Not a balanced fan

☹

$$\Phi : \mathbb{R}^n \longrightarrow \mathbb{R}^{\binom{n}{2}}$$

$$(a_1, \dots, a_n) \longmapsto (a_i + a_j)_{ij}$$





$$Q := \mathbb{R}^{\binom{n}{2}} / \operatorname{Im} \Phi$$

$\underline{\mathcal{M}_{0,4}^{trop}}$

$$v_{12} + v_{13} + v_{14} =$$

$$= (2, 2, 2, 2, 2, 2) =$$

$$= \phi(1, 1, 1, 1)$$

$$\Rightarrow v_{12} + v_{13} + v_{14} = 0 \in Q$$

Historical note:

$$
\begin{tikzcd}
Gr(2,n) \arrow[r, hook, "\text{Pl}"] & \mathbb{P}^{\binom{n}{2}-1} \\
\mathring{Gr}(2,n) \arrow[u, phantom, "\cup" sloped] \arrow[r, hook] \arrow[d] & T_{\binom{n}{2}-1} \arrow[u, phantom, "\cup" sloped] \arrow[d] \\
\mathring{Gr}(2,n)/T_{n-1} \arrow[r, hook] & T_{\binom{n}{2}-1}/T_{n-1} \\
M_{0,n} \arrow[u, phantom, "\cong" sloped] \arrow[r, hook] & T_{\binom{n}{2}-n} \arrow[u, phantom, "\cong" sloped]
\end{tikzcd}
$$

Gr

Speyer-Sturmfels

— . — Gr

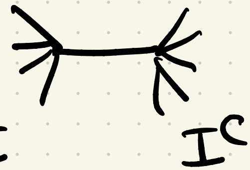
Gibney-Maclagan

$M_{0,n}^{trop}$ Tevelev

↑ TROPICALIZATION of THIS EMB INTO a TORUS

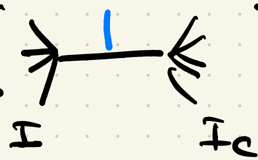
Marking on $M_{0,n}^{trop}$

Rays of $M_{0,n}^{trop}$ =



$|I| \geq 2$

$v_I :=$ dist (



)

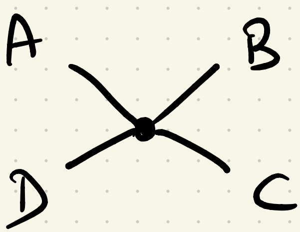
$M_{0,n}^{trop}$ is now a marked fan and each maximal cone has weight 1.

Theorem $[\mathrm{dist}(M_{0,n}^{trop})] \subseteq Q$

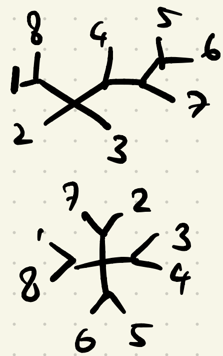
is a balanced fan, with all cones having weight 1.

PROOF: pick a codim 1 face in $M_{0,n}^{trop}$.

τ a cod 1 face corresponds to tropical curves of the form



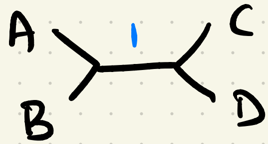
e.g.

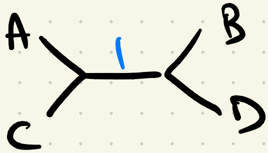


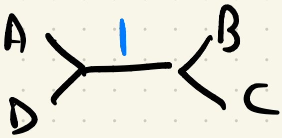
A, B, C, D : • ends
• trivalent trees

Show the case where A, B, C, D, are all trivalent trees.

Adjacent top dim cones are:

σ_1  $\rightsquigarrow$ $\sigma_{A\cup B}$

σ_2  $\rightsquigarrow$ $\sigma_{A\cup C}$

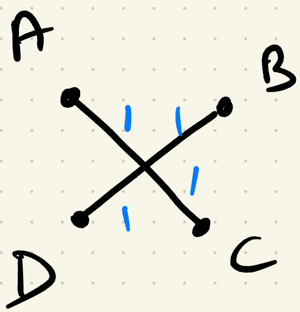
σ_3  $\rightsquigarrow$ $\sigma_{A\cup D}$

$$\boxed{\begin{array}{c} \sigma_{A\cup B} \\ + \\ \sigma_{A\cup C} \\ + \\ \sigma_{A\cup D} \\ \uparrow \\ \langle I \rangle \end{array}} \quad (\sigma_{A\cup B} + \sigma_{A\cup C} + \sigma_{A\cup D})_{ij}$$

$$\begin{array}{c} \langle I \rangle \\ \cong \\ Q \end{array}$$

- 1: if i,j belong to same trivalent tree (i.e. A) $\Rightarrow 0$
- 2: if i,j belong to different triv. trees $(i\in A, j\in B)$

$0+1+1=2$



length 0 to all other edges

$$\overset{4}{\Gamma} \in \mathcal{T}$$

$$\mathrm{dist}(\Gamma)_{ij} = \begin{cases} 0 & i,j \text{ same subset} \\ 2 & i,j \text{ different subsets} \end{cases}$$

(when some of A, B, C, D's are ends, will need to use ϕ as well)

Last time:

(1) $M_{0,n}^{trop}$ as a cone complex

(2) Forgetful morphisms

(3) $\mathrm{dist} : M_{0,n}^{trop} \longrightarrow \mathbb{R}^{\binom{n}{2}}$

$$\phi : \mathbb{R}^n \longrightarrow \mathbb{R}^{\binom{n}{2}}$$

$$Q := \mathbb{R}^{\binom{n}{2}} / \mathrm{Im}(\phi)$$

(4) Marking on $M_{0,n}^{trop}$:

$$v_I := \mathrm{dist}\left(I \succ \overset{1}{\text{—}} \prec I^c\right) \Rightarrow \omega_\Gamma = 1$$

(5) $M_{0,n}^{trop}$ is a balanced fan.

Lecture 3

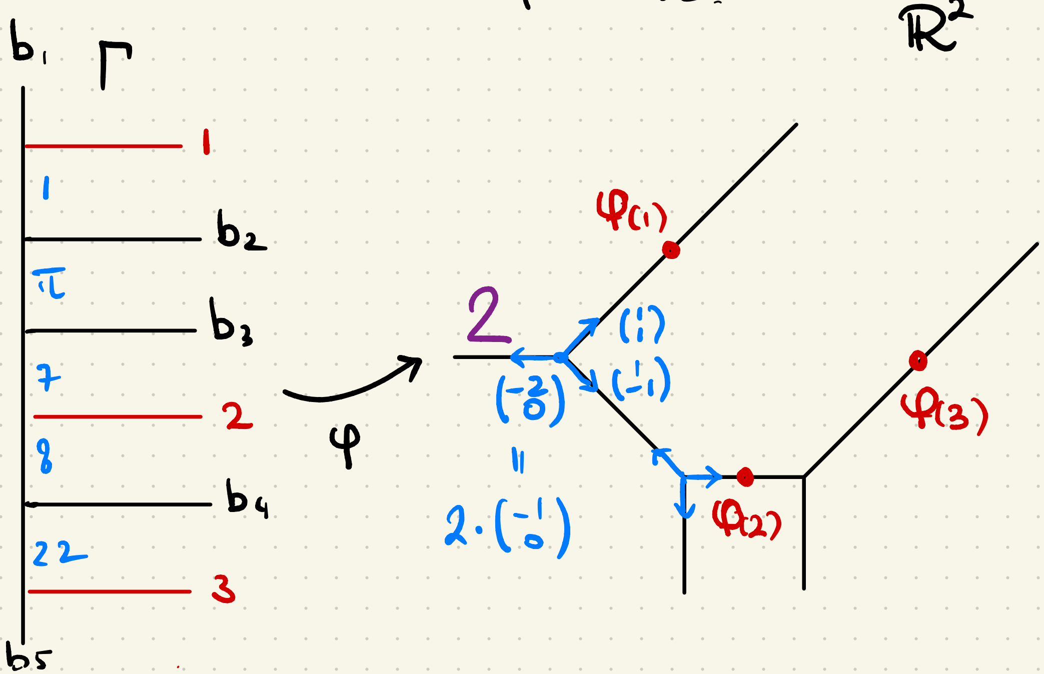
Tropical Stable Maps

Goals:

- $M^{trop}_{0,n}(\mathbb{R}^2, \Delta)$
- Embed it as a balanced fan
- Define $\overrightarrow{Ev}$ and interpret its degree as an enumerative invariant.

Tropical, rational, n-marked stable map to $\mathbb{R}^2$:

- $\Gamma \in M^{trop}_{0,n+m}$
- $\varphi: \Gamma \to \mathbb{R}^2$, $\varphi_{|e}(t) = \vec{a} + \vec{v}t$ $\quad \vec{v} \in \mathbb{Z}^2$
- direction vectors
- red ends: mapped with direction $v_i = \vec{0}$
- balancing: $\sum_{e \ni vert} \vec{v_e} = 0$
- weights: record when dir. vectors are not primitive.

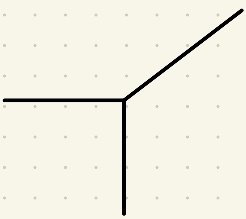


Degree of φ: multivector Δ of direction vectors of "black" ends.

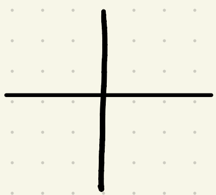
When $\Delta = \begin{pmatrix}-1\\0\end{pmatrix}^d, \begin{pmatrix}0\\-1\end{pmatrix}^d, \begin{pmatrix}1\\1\end{pmatrix}^d$

$\Rightarrow$ the map has degree d (in $\mathbb{P}^2$)

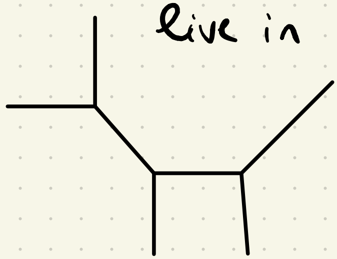
(secretely: deg of a tropical stable map tells us what toric surface it wants to live in)



$\begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}0\\-1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}$



$\begin{pmatrix}\pm 1\\0\end{pmatrix}, \begin{pmatrix}0\\\pm 1\end{pmatrix}$



$\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}0\\-1\end{pmatrix}, \begin{pmatrix}0\\-1\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$

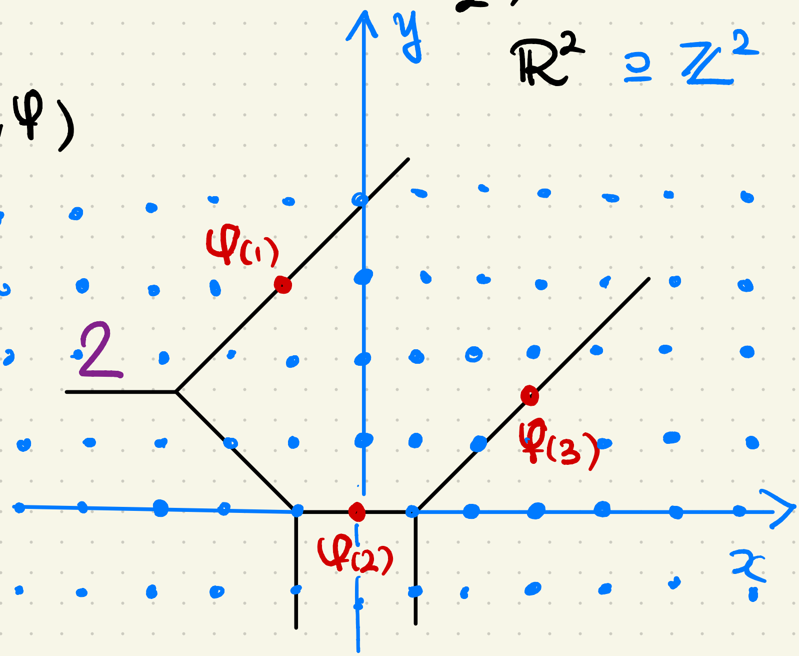
Evaluation morphisms ev_i:

$$ev_i : \mathcal{M}^{trop}_{0,n}(\mathbb{R}^2, \Delta) \longrightarrow \mathbb{R}^2$$
$$(\Gamma, \varphi) \longmapsto \varphi(i)$$

$$ev_1(\Gamma,\varphi) = (-1,3)$$
$$ev_2(\Gamma,\varphi) = (0,0)$$
$$ev_3(\Gamma,\varphi) = \left(3, \tfrac{3}{2}\right)$$

not consistent (but right idea ...)

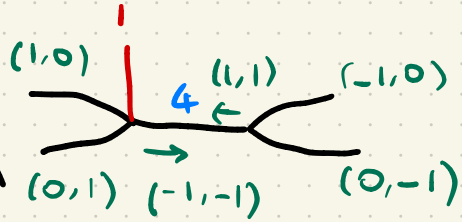
(Γ, φ)



Thm: $s \times ev_1 : \mathcal{M}^{trop}_{0,n}(\mathbb{R}^2, \Delta) \to \mathcal{M}^{trop}_{0,n+m} \times \mathbb{R}^2$

is a bijection

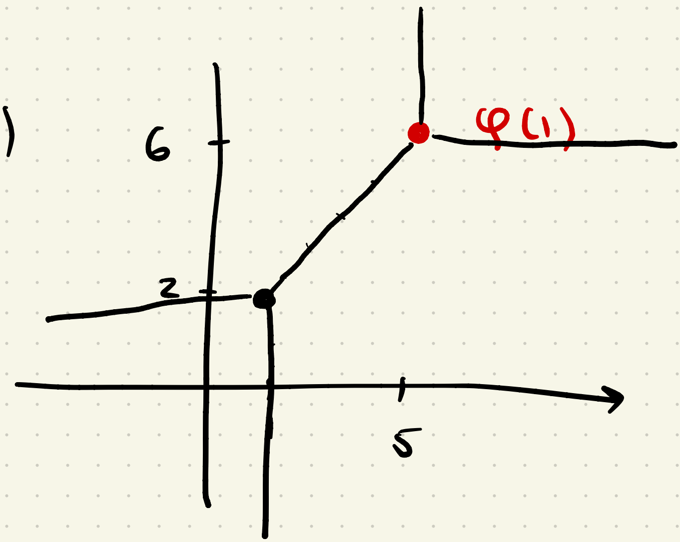
E.g.: (



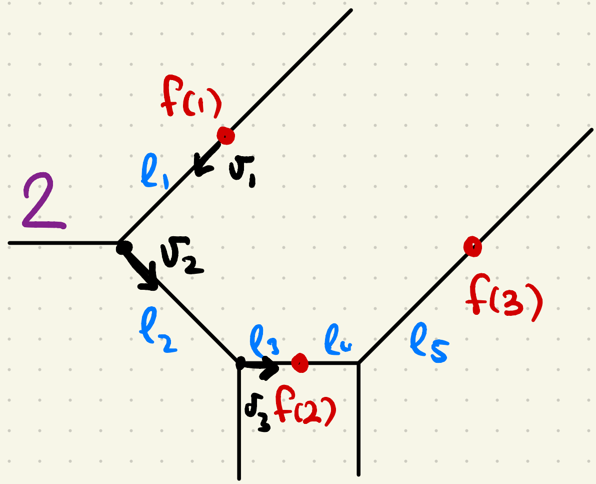
, $(5,6)$)

$\mapsto$

(Γ, φ)



$\mathbb{R}^2$



$$ev_2(\Gamma, f) = f(1) + \ell_1 \vec{v}_1 + \ell_2 \vec{v}_2 + \ell_3 \cdot \vec{v}_3$$

$$ev_3(\Gamma, f) = f(1) + \sum_i \ell_i \vec{v}_i$$

for all edges in a path from $1 \to 3$

Lemma: ev_i are restrictions of linear functions $L_i : Q \times \mathbb{R}^2 \to \mathbb{R}^2$

!! ⇒ they determine maps of fans !!

Global evaluation morphism $\vec{Ev}$

$$\vec{Ev} := ev_1 \times ev_2 \times \dots \times ev_n$$

$$\shortparallel$$

$$\mathcal{M}_{0,n}^{trop}(\mathbb{R}^2, \Delta) \longrightarrow \underbrace{\mathbb{R}^2 \times \mathbb{R}^2 \times \dots \times \mathbb{R}^2}_{n \text{ times}}$$

$\vec{Ev}$ defines a map of fans

$\vec{Ev}_*(\mathcal{M}_{0,n}^{trop}(\mathbb{R}^2, \Delta))$ is a balanced fan.

($(\mathbb{R}^2)^n$ irreducible $\Rightarrow$ if I arrange $\dim \mathcal{M}_{0,n}^{trop}(\mathbb{R}^2, \Delta) = 2n \Rightarrow \vec{Ev}$ onto)

N_Δ^{trop}

$\overset{\cdot\cdot}{=}$

$\deg \overrightarrow{Ev}$

$=$

"#" of tropical curves of degree Δ through $|\Delta| - 1$ general points in the plane!

$(n = |\Delta| - 1)$

$\uparrow$

ensures

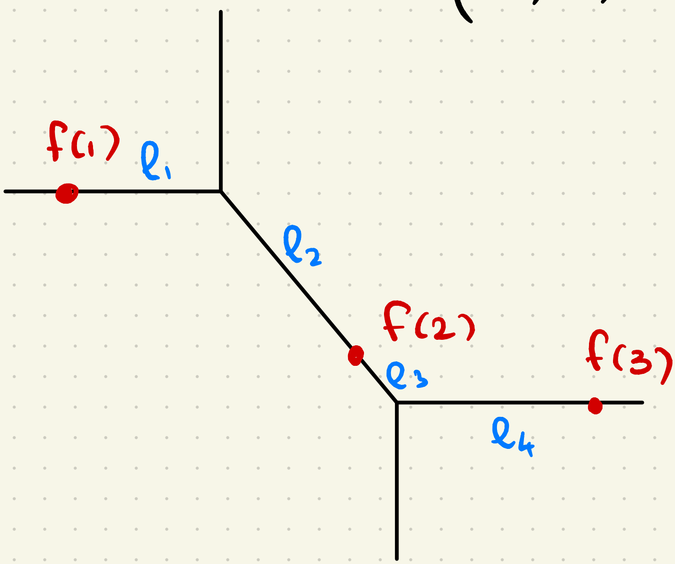
$\dim M_{0,n}^{trop}(\mathbb{R}^2, \Delta)$

$=$

$2n$

Example:

(Γ, f)



$$\deg_{\vec{Ev}}(\Gamma, f) = |\det(M)| = 1 \cdot 1 \cdot 1 = 1$$

$$M = \begin{array}{c} \\ ev_1 \\ \\ ev_2 \\ \\ ev_3 \\ \\ \end{array} \begin{array}{c} \begin{array}{cccccc} x & y & \ell_1 & \ell_2 & \ell_3 & \ell_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 & -1 & 0 \end{bmatrix} \end{array}$$

Last time:

(1) $M^{trop}_{0,n}(\mathbb{R}^2, \Delta)$

- Γ an $(n+|\Delta|)$-marked curve
- $\varphi : \Gamma \to \mathbb{R}^2$

(2) $ev_i : M^{trop}_{0,n}(\mathbb{R}^2, \Delta) \longrightarrow \mathbb{R}^2$

(3) $s \times ev_1 : M^{trop}_{0,n}(\mathbb{R}^2, \Delta) \hookrightarrow\!\!\!\!\rightarrow M_{0,n+|\Delta|} \times \mathbb{R}^2$

(4) $\vec{Ev}$ gives a map of fans

(5) If $n = |\Delta| - 1$,

$$N^{trop}_{\Delta} := \deg(\vec{Ev})$$

LECTURE 4

Mikhalkin's Correspondence Theorem

GOALS:

- Recursion among N_d^{trop}'s
- $N_d^{trop} = N_d$

Theorem (Mikhalkin)

$$\boxed{N_d^{trop} = \sum_{\substack{d_1+d_2=d \\ d_i \geq 1}} \left[\binom{3d-4}{3d_1-2} d_1^2 d_2^2 - \binom{3d-4}{3d_1-1} d_1^3 d_2 \right] N_{d_1}^{trop} \cdot N_{d_2}^{trop}}$$

Eg

$$N_1^{trop} = 1$$

$$N_2^{trop} = \left[\binom{2}{1} 1\cdot 1 - \binom{2}{2} 1\cdot 1 \right] 1\cdot 1 = 1$$

$$N_3^{trop} = \left[\binom{5}{1} 1\cdot 4 - \binom{5}{2} 1\cdot 2 \right] + \left[\binom{5}{4} 4\cdot 1 - \binom{5}{5} 8\cdot 1 \right] = 12$$

Cor: $\boxed{N_d = N_d^{trop}}$

Auxiliary map

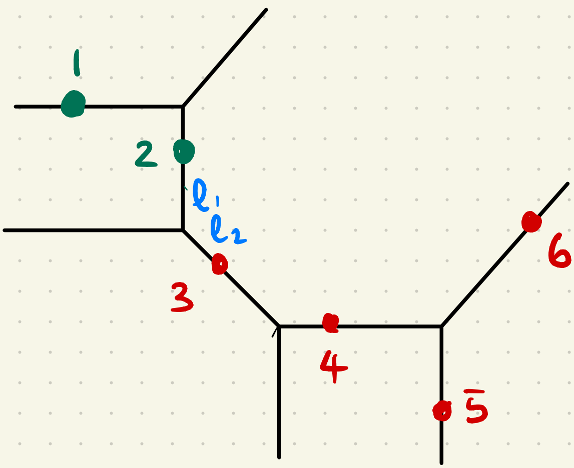
$$\cong \ \mathcal{M}^{trop}_{0,3d+3d} \times \mathbb{R}^2$$

$$\Pi : \mathcal{M}^{trop}_{0,3d}(\mathbb{P}^2, d) \to (\mathbb{R}^2)^{3d-1} \times \mathcal{M}^{trop}_{0,4}$$

$$\Pi := ev_{1,x} \times ev_{2,y} \times \underbrace{ev_3 \times \dots \times ev_{3d}}_{\text{reg. ev. morphisms}} \times \underbrace{f_4}_{\text{forgetful}}$$

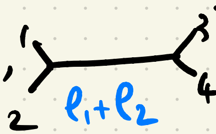
$ev_{1,x}$: evaluate x coord of 1

$ev_{2,y}$: —— y —— 2



$\mapsto$ ($x(1)$, $y(2)$, $x(3)$, $y(3)$,

$x(4)$, $y(4)$, $x(5)$, $y(5)$

$x(6)$, $y(6)$, )

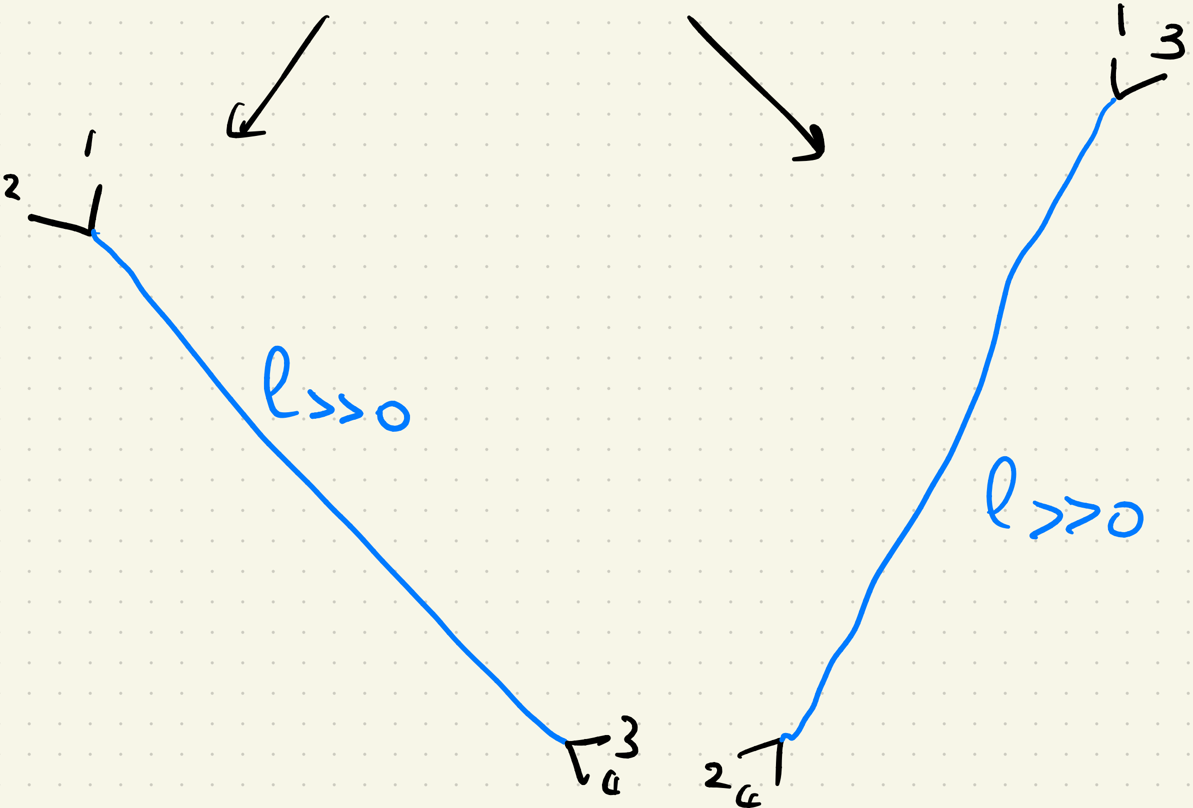
deg Π (in 2 ways)

Fix
- 2 lines

  $x = x_1$

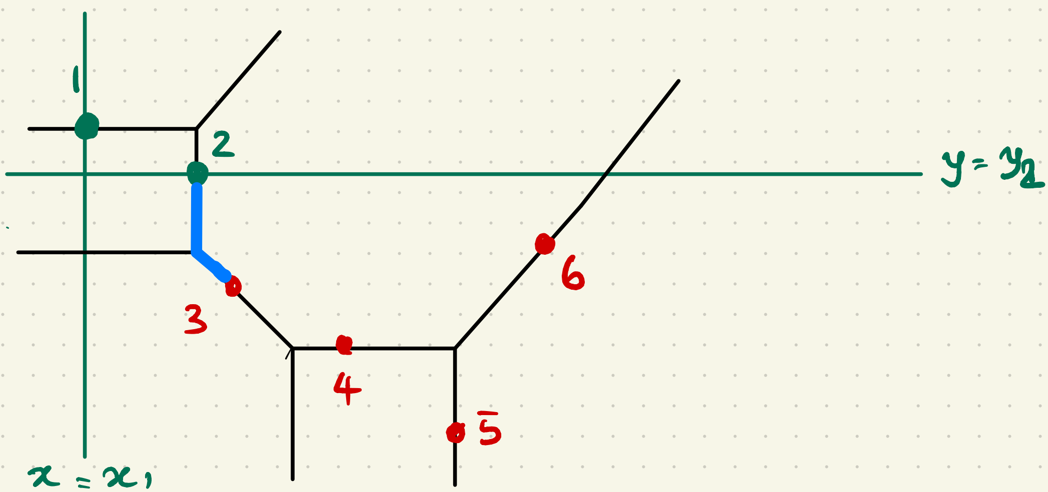
  $y = y_2$
- $3d-2$ pts

  $P_3, P_4, \ldots, P_{3d}$



① $\pi^{-1}\big(x_1, y_2, P_3, \dots, P_{3d},$  $\big)$

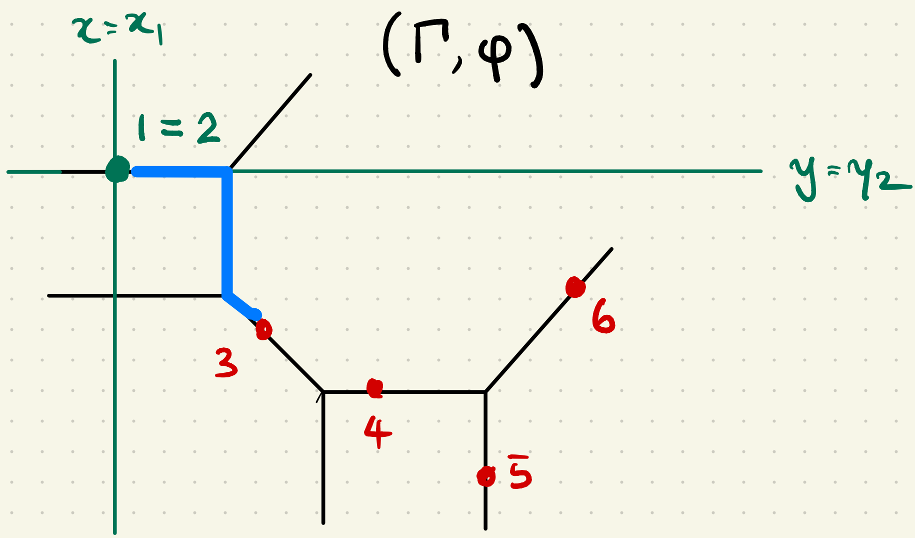
Curves in must contract some compact (internal) edge because $\ell \gg 0$



Ⓐ contracted edge adjacent to 1 & 2

These are t.s.m. that contribute to counting N_d

(Γ, φ)

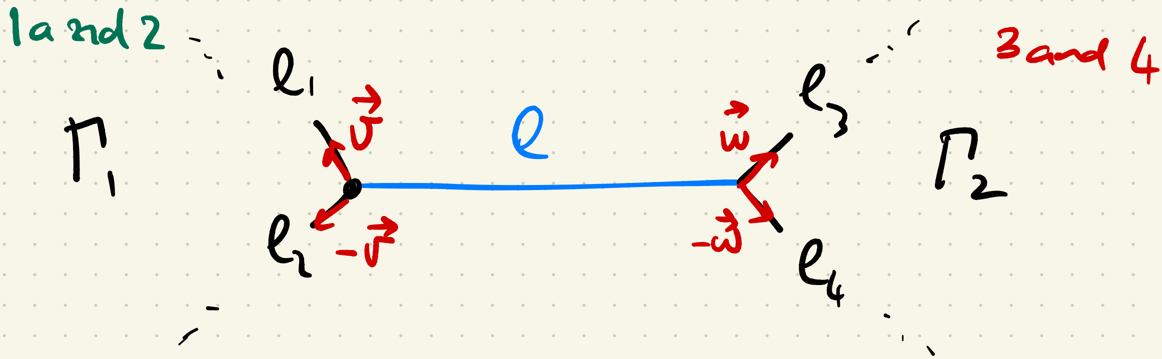


Multiplicity

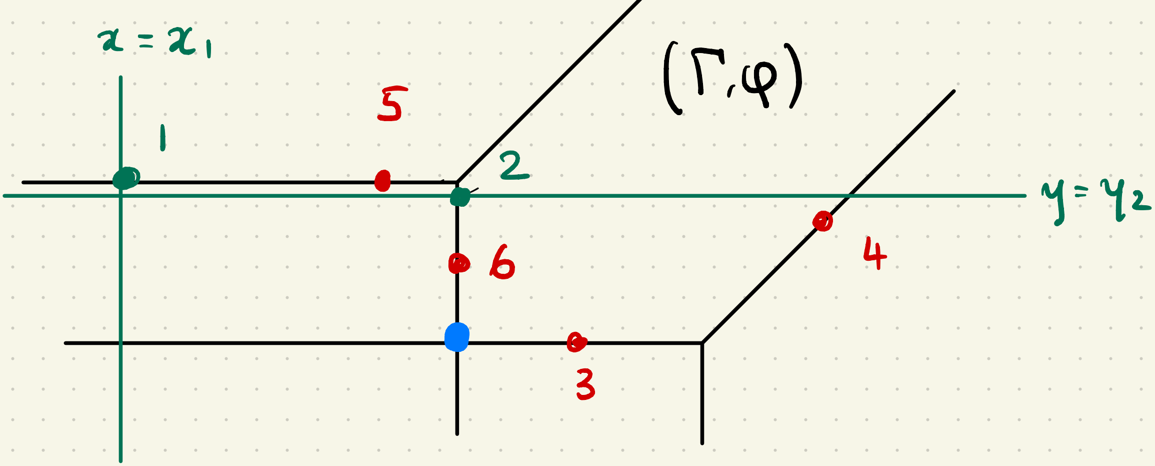
$$\mathrm{mult}_{(\Gamma,\varphi)}(\Pi) = \left|\det(M)\right|$$

$$M = \begin{array}{c} \\ ev_{1,x} \\ ev_{2,y} \\ ev_3 \\ \vdots \\ ev_{3d} \\ f_4 \end{array} \begin{array}{c} \begin{array}{cccccc} x & y & l_1 & l_2 & \cdots & l_{6d-4} \;\; l_c \end{array} \\ \left[\begin{array}{c|c} & 0 \\ & 0 \\ M_{(\Gamma,\varphi)}(E\sigma) & \vdots \\ & \vdots \\ & 0 \\ \hline \times \;\; \cdots \;\; \times & 1 \end{array}\right] \end{array}$$

(B) Contracted edge NOT adj to 1 or 2.



Removing blue edge gives rise to 2 maps (Γ_1, φ_1) (Γ_2, φ_2) of deg's d_1, d_2 with $d_1 + d_2 = d$



Lemma: $(\Gamma, \varphi) \in \pi^{-1}(x_1, y_2, P_3, \dots, P_{3d}, \gamma_1)$ in case Ⓑ:

(1) Removing contracted edge get:

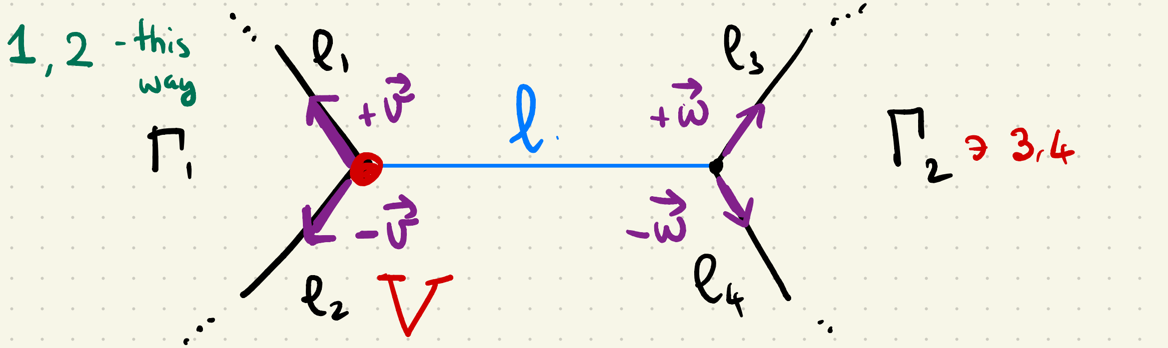
(Γ_1, φ_1) – deg d_1 $\qquad d_1 + d_2 = d$

(Γ_2, φ_2) – deg d_2

(2) $1, 2 + 3d_1 - 1$ marks $\in \Gamma_1$

$3, 4 + 3d_2 - 3$ marks $\in \Gamma_2$

(3)
$$\boxed{\begin{aligned} \mathrm{mult}_{(\Gamma,\varphi)}(\pi) = {} & \mathrm{mult}_{(\Gamma_1,\varphi_1)}(Ev_1) \cdot \\ & \mathrm{mult}_{(\Gamma_2,\varphi_2)}(Ev_2) \cdot \\ & \mathrm{mult}_{\varphi(e)}(\Gamma_1, \Gamma_2) \cdot \\ & \mathrm{mult}_{\varphi(1)}(\Gamma_1, \{x = x_1\}) \cdot \\ & \mathrm{mult}_{\varphi(2)}(\Gamma_2, \{y = y_2\}) \end{aligned}}$$



	l	lengths in Γ_1	l_1	l_2	l_3	l_4	lengths in Γ_2	$\varphi(V)$
$ev_{1,x}$	0	$*$	v_x	0	0	0	0	$\begin{matrix}1 & 0\end{matrix}$
$ev_{2,y}$	0	$*$	v_y	0	0	0	0	$\begin{matrix}0 & 1\end{matrix}$
evaluations of coordinates of points behind l_1	0	$*$	$\mathbf{v}$	0	0	0	0	$\begin{matrix}1 & 0\\ 0 & 1\end{matrix}$
evaluations of coordinates of points behind l_2	0	$*$	0	$-\mathbf{v}$	0	0	0	$\begin{matrix}1 & 0\\ 0 & 1\end{matrix}$
evaluations of coordinates of points behind l_3	0	0	0	0	$\mathbf{w}$	0	$*$	$\begin{matrix}1 & 0\\ 0 & 1\end{matrix}$
evaluations of coordinates of points behind l_4	0	0	0	0	0	$-\mathbf{w}$	$*$	$\begin{matrix}1 & 0\\ 0 & 1\end{matrix}$
f_4	1	$*$	$*$	$*$	$*$	$*$	$*$	$*$

$\Downarrow$

	$\varphi(V) - C(\tilde{l}_2, l_4 - l_3))$	lengths in Γ_1	l_1	$\tilde{l}_2 = l_2 - l_1 + \varphi(V)\mathbf{v}$	$l_3 - l_4$	l_4	lengths in Γ_2
$ev_{1,x}$	$\begin{matrix}1 & 0\end{matrix}$	$*$	v_x	0	0	0	0
$ev_{2,y}$	$\begin{matrix}0 & 1\end{matrix}$	$*$	v_y	0	0	0	0
evaluations of coordinates of points behind l_1	$\begin{matrix}1 & 0\\ 0 & 1\end{matrix}$	$*$	$\mathbf{v}$	0	0	0	0
evaluations of coordinates of points behind l_2	$\begin{matrix}1 & 0\\ 0 & 1\end{matrix}$	$*$	0	0	0	0	0
evaluations of coordinates of points behind l_3	0	0	0	$\mathbf{v}$	$\mathbf{w}$	0	$*$
evaluations of coordinates of points behind l_4	0	0	0	$\mathbf{v}$	$\mathbf{w}$	$-\mathbf{w}$	$*$

$$M_{SE} = \left[\begin{array}{cc|c|c} \vec{v} & \vec{w} & 0 & * \\ \hline \vec{v} & \vec{w} & -\vec{w} & * \end{array}\right]$$

lengths in Γ_2

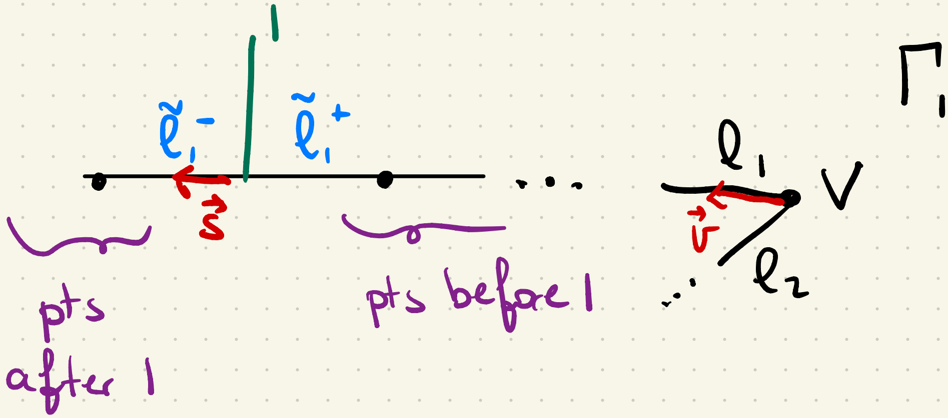
$\tilde{\ell} = \ell_3 + \ell_4$

$$M^{(E\vec{v}_2)}_{(\Gamma_2,\varphi_2)} = \left[\begin{array}{cc|c|c} 1 & 0 & 0 & * \\ 0 & 1 & & \\ \hline 1 & 0 & -\vec{w} & * \\ 0 & 1 & & \end{array}\right]$$

$$X = \left[\begin{array}{c|c} \vec{v}\ \vec{w} & 0 \\ \hline 0 & Id \end{array}\right]$$

$$M_{SE} = M_{(\Gamma_2,\varphi_2)} \cdot X$$

$$\boxed{\det M_{SE} = \mathrm{mult}_{(\Gamma_2,\varphi_2)}(E\vec{v}) \cdot \mathrm{mult}_{\varphi(e)}(\Gamma_1, \Gamma_2)}$$



$$M_{NW} = \begin{array}{c} \\ w_{1x} \\ w_{2y} \\ \text{pts bef l} \\ \\ \text{pts aft l} \\ \\ \end{array} \begin{array}{c} \begin{array}{cccccc} \varphi(V) & & \tilde{e}_1^- & \tilde{e}_1^+ & \text{others} & e_1 \end{array} \\ \left[\begin{array}{cc|c|c|c|c} \cancel{1} & \cancel{0} & \cancel{0} & s_x & \cancel{*} & \cancel{v_x} \\ 0 & 1 & \times & 0 & & v_y \\ \hline 1 & 0 & & & & \\ 0 & 1 & 0 & 0 & * & \vec{v} \\ \hline 1 & 0 & \vec{s} & 0 & * & \vec{v} \\ 0 & 1 & & & & \end{array}\right] \end{array}$$

$$\boxed{\begin{aligned} \det M_{NW} = {} & \operatorname{mult}_{\varphi(1)}(\Gamma_1, \{x = x_1\}) \cdot \\ & \operatorname{mult}_{\varphi(2)}(\Gamma_1, \{y = y_2\}) \cdot \\ & \operatorname{mult}_{(\Gamma_1, \varphi_1)}(E\vec{v}) \end{aligned}}$$

ⓒ Adding everything up

$$\deg \pi = \underbrace{N_d}_{e \text{ adj to } 1,2} +$$

$$\sum_{d_1,d_2} \underbrace{\binom{3d-4}{3d_1-1}}_{\substack{\text{choosing which points}\\ \text{belong to } \Gamma_1 \text{ and}\\ \text{which to } \Gamma_2}} \underbrace{d_1 d_2}_{\mathrm{mult}(\Gamma_1 \cdot \Gamma_2)} \cdot \overbrace{d_1}^{\Gamma_1 \cap |} \cdot \underbrace{d_1}_{\Gamma_1 \cap \text{—}} \cdot N_{d_1} \cdot N_{d_2}$$