International EMS-IMUVa school on Tropical Geometry

Abstracts of Contributed Talks

Alheydis Geiger
Eberhard Karls Universität Tübingen, Germany

Tropically counting real bitangents to smooth quartic curves

A smooth complex quartic has exactly 28 bitangent lines, while a smooth real quartic has either 4, 8, 16 or 28 real bitangent lines.

A smooth tropical quartic curve has exactly seven tropical bitangent classes. Their shapes can vary within the same combinatorial type of curve. We study deformations of these shapes, and we show that the real lifting conditions (determined by Cueto and Markwig) are independent of the deformations. Our results allow to break the existence of tropical bitangents of quartics down to an analysis of the dual triangulation. We implemented a polymake extension that enabled us to compute the possible numbers of real bitangents of quartics in polymake. With this, we deduce a tropical proof of Plücker and Zeuthen's count of the number of real bitangents to smooth plane quartic curves.

This is joint work with Marta Panizzut.

Andrés Jaramillo Puentes
Universität Duisburg-Essen, Germany

Enriched tropical intersection

Tropical geometry has been proven to be a powerful computational tool in enumerative geometry over the complex and real numbers. In this talk we present an example of a quadratic refinement of this tool, namely a proof of the quadratically refined Bézout's theorem for tropical curves. We recall the necessary notions of enumerative geometry over arbitrary fields valued in the Grothendieck-Witt ring. We will mention the Viro's patchworking method that served as inspiration to our construction based on the duality of the tropical curves and the refined Newton polytope associated to its defining polynomial. We will prove that the quadratically refined multiplicity of an intersection point of two tropical curves can be computed combinatorially. We will use this new approach to prove an enriched version of the Bézout theorem and of the Bernstein–Kushnirenko theorem, both for enriched tropical curves. Based on a joint work with S. Pauli.

Celia Gómez Galdós
Universidad de Cantabria, Spain

On real tropical singularities

A purely tropical characterization of tropical singularities of order at least k of an hypersurface is already known [1]. However, in the real case there just exists a characterization of tropical singular points [2]. We provide a characterization of real tropical singular points of order at least k of an hypersurface.

To achieve this we propose a formulation of a real tropical semiring. That is, we have worked on a generalization of the previous results, including new tools to get our goal.

[1] Alicia Dickenstein and Luis F. Tabera. Singular tropical hypersurfaces. Discrete & Computational Geometry 47, 430-453, 2012. DOI: 10.1007/s00454-011-9364-6.
[2] Luis Felipe Tabera. On real tropical bases and real tropical discriminants. Collect. Math. 66(1), 77–92, 2015.

Tara Fife
Queen Mary University of London, UK

Minimal tropical bases for Schubert matroids

Matroids grew out of a desire to abstract the notions of independence in vector spaces and independence in graphs. The circuits, or minimal dependent sets, of a matroid correspond to tropical hyperplanes of a tropical projective space. The intersection of all such hyperplanes is the Bergman fan of the matroid. A tropical basis is any subcollection of circuits such that the intersection of their corresponding hyperplanes is the Bergman fan. In general, matroids can have multiple minimal tropical bases, and describing a minimal tropical basis for a matroid can be difficult. In this talk, we will describe a collection of natural minimal tropical bases for Schubert matroids.

Nicholas Anderson
Queen Mary University of London, UK

Tropically Realizable Matroids

A matroid M is said to be tropically realizable if there is a tropical ideal I for which $V(I)$ is equal to the Bergman fan of M. Put into other words, the class of tropically realizable matroids are those (valuated) matroids whose associated tropical linear space is a realizable tropical variety. In this talk, we will see how understanding the theory of matroid symmetric products informs and is, in a way, equivalent to solving the question of realizability for tropical linear spaces with weights equal to 1.

Uriel Sinichkin
Tel Aviv University, Israel

Refined invariants and characteristic numbers in positive genus

Gottsche-Schroeter refined invariants are tropical invariants which depend on a formal parameter y, and specialize to rational descendant Gromov-Witten invariants for $y = 1$ and to broccoli invariants for $y = -1$. Schroeter and Shustin introduced a version of those invariants in genus 1, but the significance of their values at $y = 1$ and $y = -1$ have remained unknown.

In this talk we will discuss a more local description of Schroeter-Shustin invariants and present a complex algebro-geometric meaning for their values at $y = 1$. If time permits, we will discuss possible generalizations to higher genus. This talk is based on joint work with Eugenii Shustin.

Courtney George
University of Kentucky, USA

Mori dream space bundles and configurations of points on tropicalized linear spaces

Mori dream spaces have nice, predictable behavior, making them desirable spaces to have. However, there has not yet been a complete classification of which spaces are allowed to call themselves “Mori dream”. In 2019, Kaveh and Manon characterized toric vector bundles by configurations of points on tropicalized linear spaces. I will give a sufficient combinatorial condition for point configurations that gives toric vector bundles that are Mori dream space.

Desmond Coles
The University of Texas at Austin, USA

Spherical tropicalization

In this talk I will review recent developments on tropical geometry for spherical varieties. In particular, I will present new results on the connections with Berkovich geometry.

George Torres
The University of Texas at Austin, USA

Tropical and Convex Geometry of Depth Two Neural Networks

We establish sufficient criteria for a convex piecewise linear function f to be representable as a depth two ReLU neural network with positive weights. By studying the tropical and convex geometry of the Legendre transform of f, we show that the weights of said neural network are determined by a quadratic program which decouples into a series of linear feasibility problems. In addition, we provide an explicit algorithm to produce such a neural network, should it exist. As a corollary, we also provide a polynomial time algorithm for determining if two depth two ReLU neural networks with positive weights and different architectures are pointwise equal.

Angela C. Hanson
University of Kentucky, USA

Application of graph theory to the Wahl map

We will discuss the Wahl map and why we care about its surjectivity. Following the work of Ciliberto, Harris, and Miranda, we will simplify the problem by turning this into a graph theoretic problem. We will see that the rank of the Wahl map detects planarity of the graph.

Time permitting, we will discuss other graphic properties that may be illuminated by the Wahl map.

Luis Crespo Ruiz Universidad de Cantabria, Spain

Multitriangulations and tropical Pfaffians

The k-associahedron $Ass_k(n)$ is the simplicial complex of $(k+1)$-crossing-free subgraphs of the complete graph with vertices on a circle. Its facets are called k-triangulations.

We explore the connection of $Ass_k(n)$ with the Pfaffian variety $Pf_k(n) \subset K^{\binom{n}{2}}$ of anti-symmetric matrices of rank $\leq 2k$.

First, we characterize the Gröbner cone $Grob_k(n) \subset \mathbb{R}^{\binom{n}{2}}$ producing as initial ideal of $I(Pf_k(n))$ the Stanley-Reisner ideal of $Ass_k(n)$ (that is, the monomial ideal generated by $(k+1)$-crossings).

We then look at the tropicalization of $Pf_k(n)$ and show that $Ass_k(n)$ embeds naturally as the intersection of $trop(Pf_k(n))$ and $Grob_k(n)$, and is contained in the totally positive part $trop^+(Pfk(n))$ of it.

We show that for $k = 1$ and for each triangulation T of the n-gon, the projection of this embedding of $Ass_k(n)$ to the $n3$ coordinates corresponding to diagonals in T gives a complete polytopal fan, realizing the associahedron. This fan is linearly isomorphic to the $\mathbf{g}$-vector fan of the cluster algebra of type A, shown to be polytopal by Hohlweg, Pilaud and Stella in (2018).

Sebastian Debus
UiT The Arctic University of Norway

(Even) Symmetric PSD and SOS forms

In this talk we consider the so-called non-normalized limits of symmetric and even symmetric forms (homogeneous polynomials). To do so, we identify (even) symmetric forms of degree d for sufficiently many variables. The sets of positive semidefinite (non negative) and sums of squares of fixed degree form nested decreasing sequences under this identification. We completely characterize the question of non-negativity versus sums of squares in the non-normalized limit case.

We begin by examining the symmetric quartics and provide test sets for non negativity and the property of being a sum of squares for the limit forms, and present interesting examples. Then, we consider even symmetric sextics and prove that the set of all psd limit forms is not semialgebraic. Finally, we study the tropicalizations of the duals to even symmetric psd and sos forms, which we compute via tropicalizing spectrahedra and tropical convexity. Tropicalization reduces the study of even symmetric limit cones to the study of polyhedral cones.

This is joint work together with Jose Acevedo, Grigoriy Blekherman and Cordian Riener.