Tropical Ideals

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Tropicalizing linear subspaces:

Ex: $I = \langle x_1 - 2x_2 + x_3, \; -2x_1 + 4x_2 + x_4 \rangle \subseteq \mathbb{C}[x_1, \dots, x_4]$

$\downarrow$

trivial valuation

$L = V(I) \subseteq \mathbb{C}^4$ or $\mathbb{P}^3$

What is trop (L)?

$x_1 \oplus x_2 \oplus x_3 \rightsquigarrow \min(x_1, x_2, x_3)$ att. twice.

$x_1 \oplus x_2 \oplus x_4 \rightsquigarrow \min(x_1, x_2, x_4)$ att twice.

⚠ $\underline{2x_3 + x_4} \in I$

$x_3 \oplus x_4 \rightsquigarrow \min(x_3, x_4)$ att. twice

<u>Theorem</u> If I is generated by linear polynomials then the polynomials of degree 1 in I form a <u>tropical basis</u>, i.e.

$$\operatorname{trop}(V(I)) = \bigcap_{\substack{f \in I \\ f \text{ of degree } 1}} \operatorname{trop}(V(f))$$

In fact, you can take linear polynomials of minimal support $\mathrm{supp}\left(\sum \alpha_i x_i\right) = \{x_i \mid \alpha_i \neq 0\}$

<u>Ex. continued:</u>

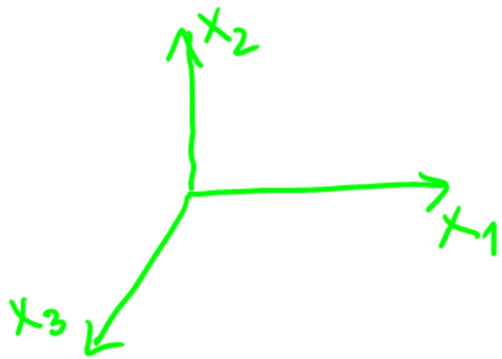
trop(L) looks like

$x_3 = x_4$

$\min(x_1, x_2, x_3)$ att. twice.

Picture in $\mathbb{R}^4 / \mathbb{R} \cdot \mathbb{1} \cap \{x_3 = x_4\}$

$\mathbb{1} \to (1,1,1,1)$.



In $\mathbb{R}^4$ it looks like

Matroids:

If $L \subseteq K[x_1, \dots, x_n]$ is a linear ideal, the *matroid* $M(L)$ has *circuits*

$$\mathcal{C} \subseteq 2^{\{1,\dots,n\}}$$

$$\mathcal{C} := \{ \operatorname{supp}(\ell) \mid \ell \in I \text{ is linear and has minimal support} \}$$

Ex continued:

The circuits of the matroid of the previous linear subspace are

$$\mathcal{C} = \{\{1,2,3\}, \{1,2,4\}, \{3,4\}\}$$

$$= \{123, 124, 34\}.$$

These circuits satisfy some properties:

(MC0) $\emptyset \notin \mathcal{C}$

(MC1) If $C_1, C_2 \in \mathcal{C}$ and $C_1 \subseteq C_2$ then $C_1 = C_2$.

(MCE) If $C_1, C_2 \in \mathcal{C}$ and $i \in C_1 \cap C_2$ then there exists $C_3 \in \mathcal{C}$ s.t. $i \notin C_3$ and $C_3 \subseteq C_1 \cup C_2$.

<u>Def</u>: A <u>matroid</u> is a collection $\mathcal{C} \subseteq 2^{\{1,\dots,n\}}$ satisfying (MC0), (MC1), (MCE).

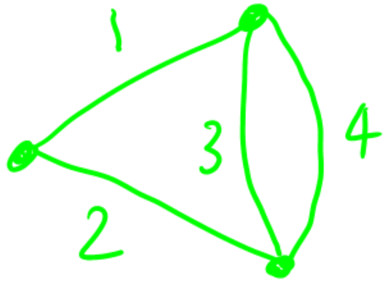
Ex: Any linear subspace L gives rise to a matroid $M(L)$. These are called realizable (or representable) matroids.

Not all matroids are realizable.

Ex: If $G = (V, E)$ is a graph. The collection $\mathcal{C} \subseteq 2^E$ defined as

$$\mathcal{C} = \{ C \subseteq E \mid C \text{ is a non-self-intersecting cycle} \}$$

is a matroid, called a *graphical matroid*.

Ex: 

$\rightsquigarrow \mathcal{C} = \{123, 124, 34\}$.

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A *dependent* set in a matroid is a subset of $\{1, \dots, n\}$ that contains a circuit. The subsets of $\{1, \dots, n\}$ that are not dependent are called *independent*.

A *basis* is a maximal independent set.

Ex: $\mathcal{B} = \{12, 13, 23, 14, 24\}$

↳ bases of our previous matroid.

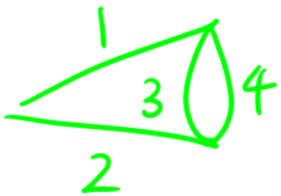
Thm: The bases of a matroid all have the same cardinality.

This is called the rank of the matroid.

Def: A flat in a matroid M is a subset $F \subseteq \{1, \dots, n\}$ such that

$$|C \setminus F| \neq 1 \quad \text{for all circuits } C \in \mathcal{C}.$$

Think of flats as "closed" subsets.

__Ex:__ $M\Big($ 

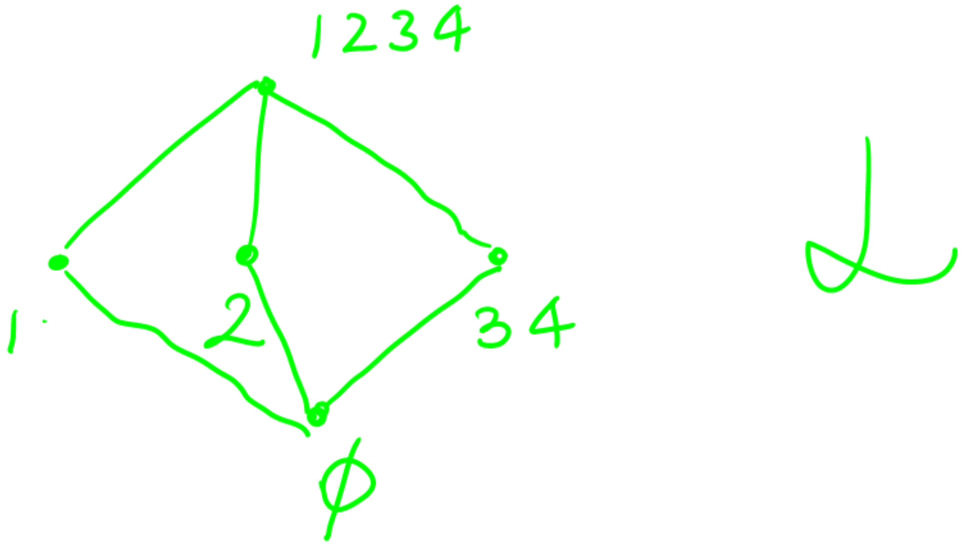
$\Big)$

$\mathcal{F} = \{\emptyset, 1, 2, 34, 1234\}.$

flats

__Def:__ The lattice of flats of M is the partial order of flats ordered by inclusion.

Ex:



Thm: The lattice of flats is always ranked, its a lattice, atomic, semimodular $\left(r(a \vee b) + r(a \wedge b) \leq r(a) + r(b)\right)$

(all this is called a geometric lattice).