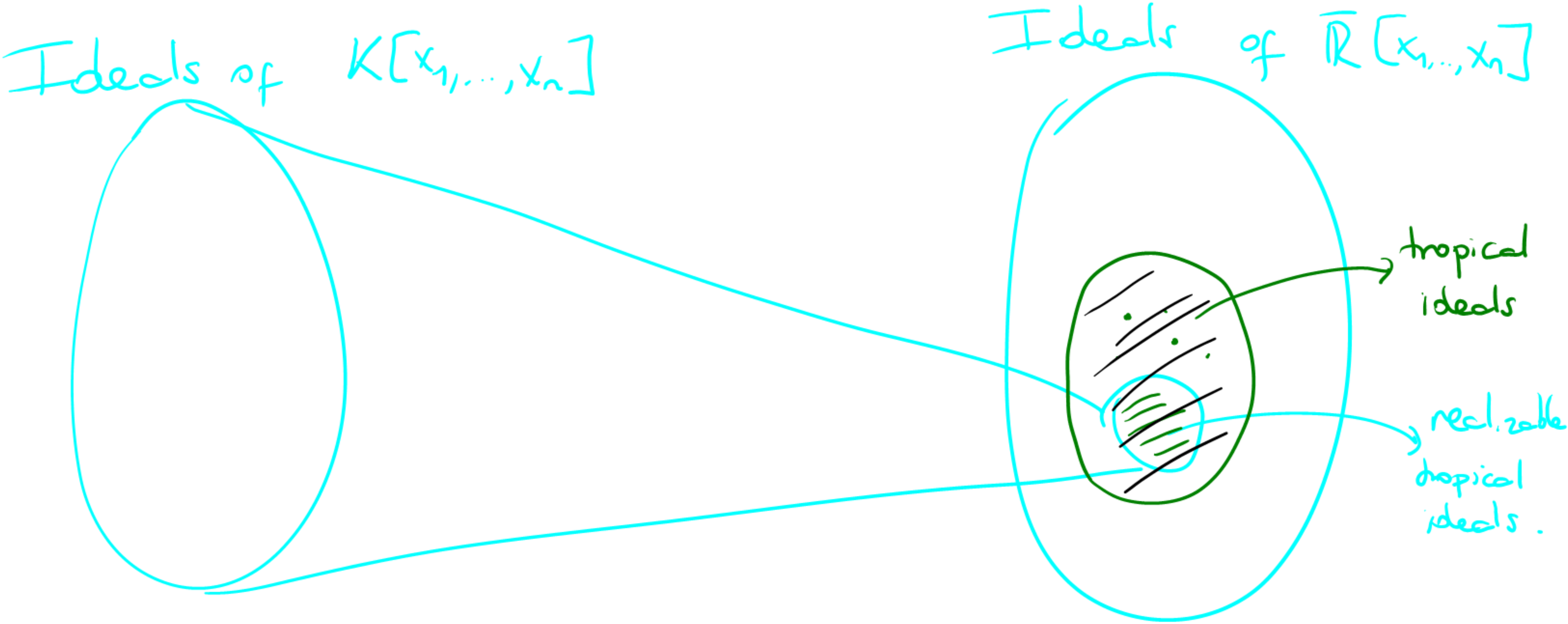
Tropical Ideals Day 4



Tropical ideal $I \subseteq \overline{\mathbb{R}}[x_1,\dots,x_n]$ if it satisfies the <u>monomial elimination axiom</u>:

- If $f, g \in I$ and $[f]_{x^u} = [g]_{x^u}$ then

$\exists\, h \in I$ s.t. $[h]_{x^v} = \infty$ and $\forall\, x^v$

$$[h]_{x^v} \geq \min\left([f]_{x^v}, [g]_{x^v}\right) \text{ with equality if } [f]_{x^v} \neq [g]_{x^v}$$

<u>Def</u>: The Hilbert function of a homogeneous tropical ideal $I \subseteq \overline{\mathbb{R}}[x_1, \dots, x_n]$ is the function

$$H_I : \mathbb{Z}_{\geq 0} \longrightarrow \mathbb{Z}_{\geq 0}$$

$$H_I(d) = \text{codimension of } I_d \text{ in } \overline{\mathbb{R}}^{\mathrm{Mon}_d}.$$

$= \text{rank of the valuated matroid } \mathcal{M}(I_d)$

$= \text{max size of a } B \subseteq \mathrm{Mon}_d \text{ s.t}$

$B \notin \mathrm{supp}(f)$ for any $f \in I$.

If $I = \mathrm{trop}(J)$ then $H_I = H_J = \dim \left(K[x_1,\dots,x_n]/J \right)_d$

$\rightarrow$ ideal in $K[x_1, \dots, x_n]$

<u>Theorem</u>: H_I is eventually a polynomial, called the Hilbert polynomial of I.

<u>Theorem</u>: There is no infinite ascending chain

$$I_1 \subsetneq I_2 \subsetneq \cdots \subseteq \overline{\mathbb{R}}[x_1, \dots, x_n]$$

of tropical ideals.

Initial ideals:

If $f \in \overline{\mathbb{R}}[x_0, \dots, x_n]$ is homogeneous and $w \in \mathbb{R}^{n+1}$ then $f = \bigoplus c_u \odot x^u$

$$\mathrm{in}_w(f) = \bigoplus_{\substack{c_u + u \cdot w \\ \text{is minimal}}} x^u$$

(monomials that achieve the min in $f(w)$)

If $I \subseteq \overline{\mathbb{R}}[x_0, \dots, x_n]$

$$\mathrm{in}_w(I) = \langle \mathrm{in}_w(f) \mid f \in I \text{ homogeneous.} \rangle$$

Key fact: If I is a tropical ideal then $\mathrm{in}_w(I)$ is also a tropical ideal and $H_{\mathrm{in}_w(I)} = H_I$.

- **Proof idea for H_I is polynomial**

I tropical ideal. Pick generic $w \in \mathbb{R}^{n+1}$.

$\mathrm{in}_w(I)$ is a monomial ideal.

$H_I = H_{\mathrm{in}_w(I)} \leftarrow$ eventually polynomial.

Def: The **Gröbner complex** of a tropical ideal I is the partition of $\mathbb{R}^{n+1}$ according to what $\mathrm{in}_w(I)$ is.

Thm: The Gröbner complex of a tropical ideal is a **finite polyhedral complex**.

Note:

$$V(I) = \left\{ w \in \mathbb{R}^{n+1} \;\middle|\; \begin{array}{c} \mathrm{in}_w(I) \text{ contains} \\ \text{no monomials} \end{array} \right\}$$

Cor:

$V(I)$ is a finite polyhedral complex.

Thm: $V(I)$ comes with natural multiplicities on its maximal cones. It is a balanced polyhedral complex.

Question: Is every balanced polyhedral complex of the form $V(I)$ for I tropical ideal?

__Answer__: No. There is no tropical ideal I such that $V(I) = \text{Bergman}(V_8 \oplus U_{2,3})$

(this boils down to non-existence of tensor product of matroids).

__Thm__: $V(I)$ is a d-dimensional polyhedral complex where d is the degree of the Hilbert polynomial.

__Open problem__: Find useful algebraic/combinatorial conditions on I that ensures $V(I)$ is pure.

Open problem: Commutative algebra for tropical ideals: sum, intersection, free resolutions, primary decomposition, ...

Open problem: How can we finitely present a tropical ideal?