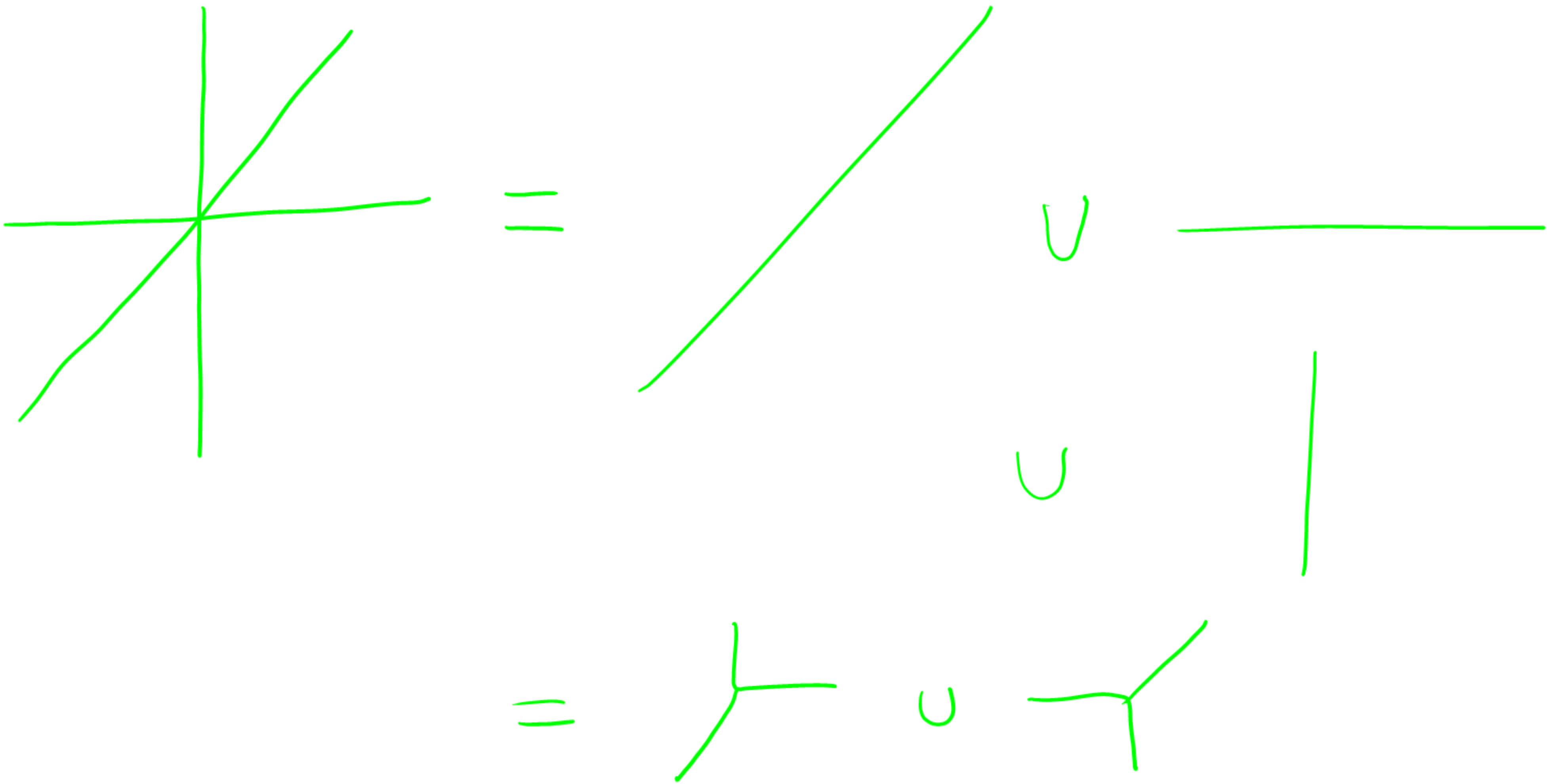
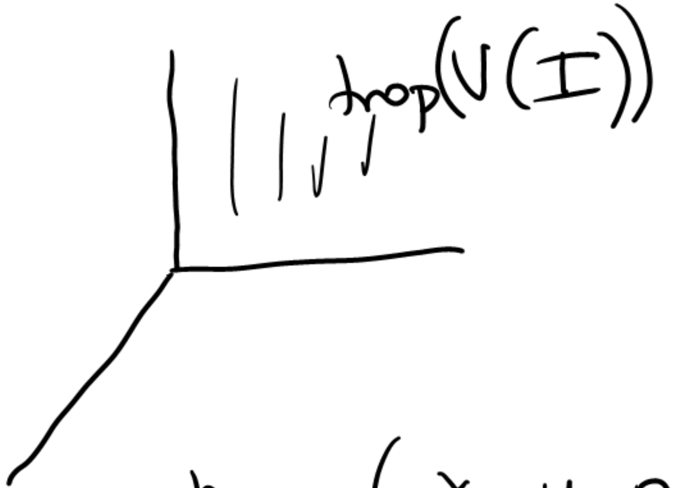
Tropical Ideals Day 3

Algebraic foundations for tropical geometry?



Jeff and Noah Giansiracusa:

Ex: $I = \langle x+y+1 \rangle$



$\min(x, y, 0)$ attained twice.

In $\mathrm{trop}(V(I))$ $\quad \underline{\underline{x} \oplus \underline{y} \oplus 0} = \underline{x \oplus y} = \underline{x \oplus 0}$

$= \underline{y \oplus 0}$.

as piecewise linear functions.

In classical algebraic geometry,

$$\mathbb{R}[\vec{x}]/I \qquad f \sim g \iff f - g \in I$$

For semirings, we need <u>congruences</u>.

<u>Def</u>: A <u>congruence</u> on $\overline{\mathbb{R}}[x_1, \dots, x_n]$ is an equivalence relation such that

$$f_1 \sim g_1 \text{ and } f_2 \sim g_2 \implies \begin{array}{l} f_1 \oplus f_2 \sim g_1 \oplus g_2 \\ f_1 \odot f_2 \sim g_1 \odot g_2. \end{array}$$

Def: The <u>bend congruence</u> of a tropical variety $\operatorname{trop}(V(J))$ is given by

$$\operatorname{bend}(\underset{\text{tropical polynomial}}{f}) = \{ f \sim f_{\hat{v}} : v \in \operatorname{supp}(f) \}$$

$$\left(\text{If } f = \bigoplus c_u \odot x^u, \quad f_{\hat{v}} = \bigoplus_{u \neq v} c_u \odot x^u\right)$$

$$\underbrace{\operatorname{bend}(\operatorname{trop}(I))}_{\text{algebraic structure associated to } \operatorname{trop}(V(I)).} = \langle \operatorname{bend}(\operatorname{trop}(f)) : f \in I \rangle$$

The coordinate semiring of $\operatorname{trop}(V(I))$

is $\overline{\mathbb{R}}[x_1, \dots, x_n] \big/ \operatorname{bend}(\operatorname{trop}(I))$

Ex:

$I = \langle x + y + 1 \rangle$

$x \sim X$

$$\operatorname{bend}(\operatorname{trop}(I)) = \Big\langle \begin{aligned} & x \oplus y \oplus 0 \sim x \oplus y \\ & \qquad \sim x \oplus 0 \\ & \qquad \sim y \oplus 0 \\ & x^2 \oplus yx \oplus x \sim x^2 \oplus yx \\ & \qquad \sim x^2 \oplus x \\ & \qquad \sim yx \oplus x \\ & \qquad \vdots \end{aligned} \Big\rangle$$

In $I \ni x^2 + \cancel{xy} + x - \cancel{yx} - y^2 - y$

In $\mathrm{bend}(\mathrm{trop}(I)) \ni \{ x^2 \oplus x \oplus y^2 \oplus y \sim$

$x^2 \oplus x \oplus y^2 \sim \dots \}$

<u>Prop</u>: $\mathrm{bend}(\mathrm{trop}(I))$ determines $\mathrm{trop}(V(I))$

$$\mathrm{trop}(V(I)) = \mathrm{Hom}\left(\overline{\mathbb{R}}[x_1, \dots, x_n] / \mathrm{bend}(\mathrm{trop}(I)), \overline{\mathbb{R}} \right)$$

This is true even with multiplicities.

Theorem: $\operatorname{bend}(\operatorname{trop}(I))$ and $\operatorname{trop}(I)$ contain the same information.

$$\operatorname{trop}(I) := \underbrace{\{\operatorname{trop}(f) : f \in I\}}_{\text{an ideal in } \overline{\mathbb{R}}[x_1, \dots, x_n]}.$$

⚠ $\overline{\mathbb{R}}[x_1, \dots, x_n]$ is not Noetherian:

$$\operatorname{trop}(\langle x - y \rangle) = \{\underline{x \oplus y},\ x^2 \oplus xy,\ xy \oplus y^2,\ x^2 \oplus y^2, \dots\}$$

$$= \langle x \oplus y,\ x^2 \oplus y^2,\ x^3 \oplus y^3,\ \dots \rangle$$

- Varieties of arbitrary ideals in $\overline{\mathbb{R}}[x_1, \ldots, x_n]$ might not be polyhedral complexes.

- <u>Idea</u>: Restrict to a class of ideals in $\overline{\mathbb{R}}[x_1, \ldots, x_n]$ that "look like" trop (I).

<u>Def</u>: An ideal I in $\overline{\mathbb{R}}[x_1, \ldots, x_n]$ is called a <u>tropical ideal</u> if

$$I_{\leq d} = \{ f \in I \mid \deg(f) \leq d \}$$

is a tropical linear space in $\overline{\mathbb{R}}^{\underbrace{\mathrm{Mon}_{\leq d}}}$

monomials of deg $\leq d$.

<u>Equivalently</u>, I satisfies the <u>monomial elimination axiom</u>:

coeff. of f at monomial x^u

- If $f, g \in I$ and $[f]_{x^u} = [g]_{x^u}$ $\exists\, h \in I$ such that $[h]_{x^u} = \infty$ and for any other x^v

$$[h]_{x^v} \geqslant \min\left([f]_{x^v}, [g]_{x^v}\right).$$

and this is an equality if $[f]_{x^v} \neq [g]_{x^v}$

<u>Ex:</u> • $\operatorname{trop}(I)$ is a <u>tropical ideal</u>

for any $I \subseteq K[x_1, \dots, x_n]$

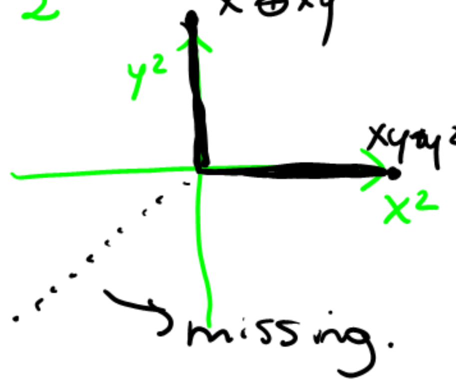
(think these are the realizable tropical ideals).

• $I = \langle x \oplus y \rangle \subseteq \overline{\mathbb{R}}[x,y]$ is <u>not</u> a tropical ideal because

$x^2 \oplus xy \,,\; xy \oplus y^2 \in I$

$\Longrightarrow \; x^2 \oplus y^2 \in I$ contradiction

$I_2 \subseteq \overline{\mathbb{R}}^{\mathrm{Mon}_2}$



• __Ex:__ A non-realizable tropical ideal.

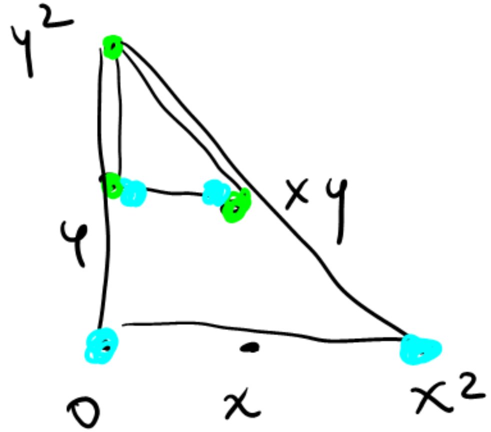
$I \subseteq \overline{\mathbb{R}}[x,y]$ is generated by

$$f = \bigoplus_{v \in C} x^v$$ where C

is a collection of $k+2$ monomials

in a standard k-triangle (and C minimal with this property).

$\mathrm{Mon}_{\leq 1}$



$\mathrm{Mon}_{\leq 2}$

$0 \oplus x \oplus y \in I$

$y \oplus xy \oplus y^2 \in I$

$0 \oplus x^2 \oplus y \oplus xy \in I$