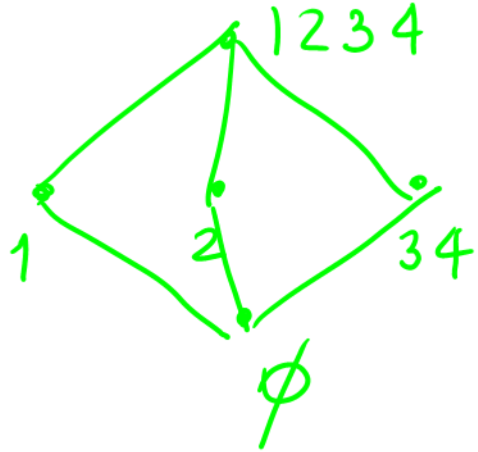
Tropical Ideals Day 2

M()

$$\mathcal{C} = \{123,\ 124,\ 34\}$$

$$\mathcal{B} = \{12,\ 13,\ 14,\ 23,\ 24\}$$

$\mathcal{F} =$ 

F is a flat if

$|C \setminus F| \neq 1$ for all $C \in \mathcal{C}$

<u>Theorem:</u> Let L be a linear subspace of K^n. A point $w \in \mathbb{R}^n$ is in $\mathrm{trop}(L)$ if and only if when you order the coordinates of w

C

$$w_{i_1} = w_{i_2} = w_{i_3} < w_{i_4} = w_{i_5} = w_{i_6} < \cdots < w_{i_{n-1}} = w_{i_n}$$

F_1

F_2

F_{k-1}

F_k

all the F_is are flats of $M(L)$.

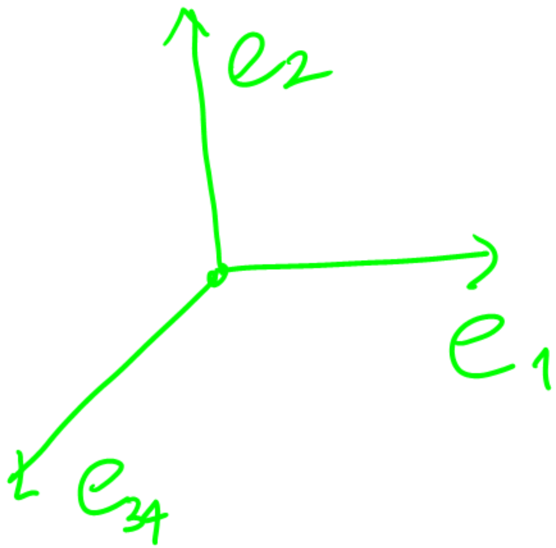
<u>Reformulation:</u> $\operatorname{trop}(L) \subseteq \mathbb{R}^n/\mathbb{R}\cdot\mathbf{1}$ is the fan with rays $e_F = \sum_{i \in F} e_i$ for F a flat of $M(L)$, and faces

$$\mathbb{R}_{\geq 0}(e_{F_1}, e_{F_2}, \dots, e_{F_k}) \quad \text{for } F_1 \subseteq F_2 \subseteq \dots \subseteq F_k$$

pos

and $F_i \neq \emptyset$, $F_i \neq \{1, \dots, n\}$

<u>Ex:</u> In the previous matroid, $\operatorname{trop}(L)$ looks like



in $\mathbb{R}^4/\mathbb{R}\cdot\mathbb{1}$

$\longrightarrow$ Bergman fan of $M(L)$.

<u>Remark</u>: This construction makes sense even for non-realizable matroids!

Matroids are the linear algebra of tropical geometry.

The Bergman fan of a matroid is a realization of the "order complex" of the lattice of flats.

This implies that

$$\underbrace{\operatorname{trop}(L)}_{\subseteq \mathbb{R}^{n-1}} \cap S^{n-2} \text{ is homotopic}$$

to a wedge of spheres.

Tropicalizing linear subspaces under a non-trivial valuation.

Ex: $I = \langle x_1 - 2x_2 + x_3,\ -2x_1 + 4x_2 + x_4 \rangle \subseteq \overline{\mathbb{Q}}[x_1, \dots, x_4]$

$\downarrow$

$\mathrm{val} = \mathrm{val}_2$.

$$\left.\begin{array}{l} x_1 \oplus 1x_2 \oplus x_3 \\[2em] 1x_1 \oplus 2x_2 \oplus x_4 \\[2em] 1x_3 \oplus x_4 \end{array}\right\} \text{ attained at least twice}$$

The structure is determined by

the corresponding <u>valuated matroid</u>.

<u>Def</u>: The (valuated) circuits of a valuated matroid are a collection $\mathcal{C}$ of vectors in $\overline{\mathbb{R}}^n$ satisfying:

(VC0) $(\infty, \infty, \dots, \infty) \notin \mathcal{C}$.

(VC1) If $c, d \in \mathcal{C}$ and $\underset{\{i \in \{1,\dots,n\} \mid c_i \neq \infty\}}{\underset{\shortparallel}{\mathrm{supp}(c)}} \subseteq \mathrm{supp}(d)$

then $\mathrm{supp}(c) = \mathrm{supp}(d)$.

(VC2) If $c \in \mathcal{C}$ then $c + \lambda(1,1,\dots,1) \in \mathcal{C}$ for any $\lambda \in \mathbb{R}$.

(VCE) If $c, d \in \mathcal{C}$ and $c_i = d_i \neq \infty$ then $\exists\, e \in \mathcal{C}$ s.t. $e_i = \infty$ and $\forall\, j \neq i \quad e_j \geq \min(c_j, d_j)$.

• __Def__ : The __tropical linear space__ associated to a valuated matroid $\mathcal{C}$

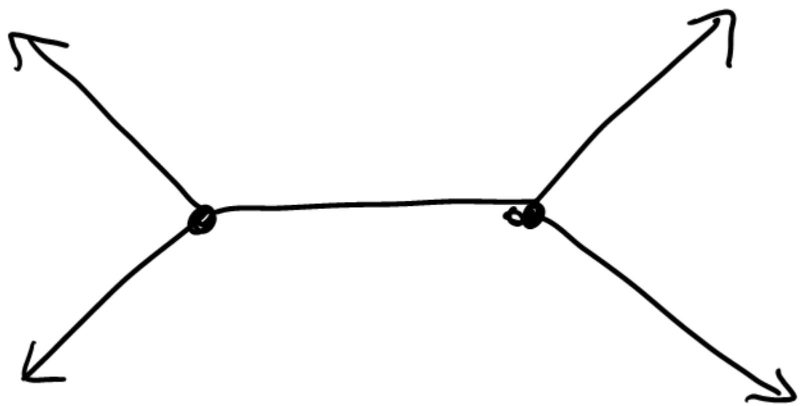
is

$$\ell = \left\{ x \in \overline{\mathbb{R}}^n \;\middle|\; \min_{i \in \{1,\dots,n\}} (x_i + c_i) \text{ attained at least twice for all circuits } c \in \mathcal{C} \right\}.$$

This makes sense even if $\mathcal{C}$ is not realizable (doesn't arise from a subspace $L \subseteq K^n$).

<u>Ex:</u> $L = \text{rowspace}\begin{pmatrix} 1 & 0 & 1 & t \\ 0 & 1 & 1 & t^2 \end{pmatrix} \subseteq \mathbb{C}\{\{t\}\}^4$

trop(L) looks like



in $\mathbb{R}^4/\mathbb{R}\cdot\mathbb{1}$.

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Thm: Tropical linear spaces are pure balanced polyhedral complexes.

Thm: Let $\mathcal{L} \subseteq \mathbb{R}^n$ be a balanced polyhedral complex. TFAE:

- $\mathcal{L}$ is a tropical linear space
- $\mathcal{L}$ is closed under tropical linear combinations and all weights are 1
- $\mathcal{L}$ has degree $\underline{1}$.