Introduction to Tropical Geometry IV

Diane Maclagan

Valladolid school

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Last time: Details of proofs/techniques for the structure theorem.

Today: Other perspectives on $M_{0,n}^{trop} = trop(M_{0,n})$ and other techniques for tropicalization

$M_{0,n}$ as a linear space

(connection with Felipe's course)

Recall: $M_{0,n} = \{n \text{ distinct pts on } \mathbb{P}^1\}/\mathrm{Aut}(\mathbb{P}^1)$

$= (\mathbb{P}^1 \setminus \{0,1,\infty\})^{n-3} \setminus \text{diagonals}$

use $\mathrm{Aut}(\mathbb{P}^1)$ to take p_1, p_2, p_3 to $0, 1, \infty$

pts are distinct

$= (\mathbb{C}^* \setminus \{1\})^{n-3} \setminus \{x_i = x_j\}$

$= \mathbb{P}^{n-3} \setminus \{x_i = 0, x_i = x_j : 0 \leq i, j \leq n\}$

hyperplane arrangement

$$M_{0,n} = \mathbb{P}^{n-3} \setminus \{x_i = 0,\ x_i = x_j : 0 \leq i,j \leq n\}$$

Embed into torus:

$$x \longmapsto [x_0 : \quad : x_n : x_0 - x_1 : \quad \cdots : x_{n-1} - x_n] \in \mathbb{P}^N$$

image is in $(\mathbb{C}^*)^N$

$$N = \binom{n-2}{2} + n-2-1$$

$$= \binom{n-1}{2} - 1 = \binom{n}{2} - n$$

The image is a linear space, cut out by kernel elements of

$$\begin{pmatrix} I & \begin{matrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{matrix} & \cdots & \begin{matrix} 0 \\ \vdots \\ 0 \\ 1 \\ -1 \end{matrix} \end{pmatrix}$$

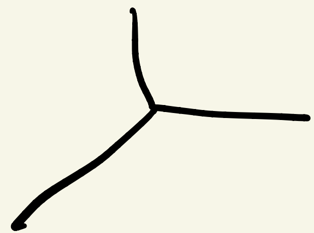
eg $\mathcal{M}_{0,4} \hookrightarrow \mathbb{P}^2$

$\mathcal{M}_{0,4} = \mathbb{P}^1 \setminus \{0,1,\infty\} \longrightarrow [x_0 : x_1 : x_0 - x_1]$

$\{[x_0 : x_1]\} \setminus \begin{matrix} x_0 = 0, \\ x_1 = 0 \\ x_0 = x_1 \end{matrix}$

Image is $V(z_2 + z_1 - z_0) \subseteq (\mathbb{C}^*)^2 \cong (\mathbb{C}^*)^3 / \mathbb{C}^*$

Tropicalization:



This is the tropicalization of a linear space!!

Its matroid is graphic — it is the matroid of K_3

In general: $\operatorname{trop}(\mathcal{M}_{0,n}) = \operatorname{trop}(K_4$ graphic matroid$)$.

Geometric Tropicalization

Rough theorem trop(X) is the image of divisional valuations on X.

Warm-up Fix an irreducible variety X.

later will be $\subseteq (K^*)^n$

The function field $K(X)$ of X is the fraction field of the coordinate ring of any affine chart.

eg $X = V(x+y-1) \subseteq (K^*)^2$ $\quad K[X] = K[x^{\pm 1}, y^{\pm 1}] / \langle x+y-1 \rangle$

$K(X) = \mathrm{frac}(\;"\;) \simeq K(x)$.

A variety Y is birational to X if there are open sets $U \subseteq X$, $V \subseteq Y$ with $U \simeq V$.

If Y is birational to X, then
$$k(Y) \simeq k(X).$$

eg $X \subseteq \mathbb{P}^n$ $\quad Y = X \cap (k^{\times})^n$ (example we'll use ~.)

X is birational to Y.

Defn Fix an irreducible variety X, and a (normal $\mathbb{Q}$-factorial) variety Y that is birational to X.

A prime divisor D on Y determines a

irreducible codim-one subvariety

divisional valuation on $k(X)$ as follows:

On an affine chart $U \subseteq Y$, write $D = V(f)$. Then $k[U]_f$ is a DVR with fraction field $k(Y) \cong k(X)$.

$\mathbb{Q}$-factorial assumption

normal assumption

$U \subseteq X$ open, $D = V(f)$, $K[U]_f$ DVR.

local with maximal ideal $\langle \pi \rangle$

Thus there is a valuation

$\mathrm{val}_D : K(Y) \cong K(X) \to \mathbb{R} \cup \infty$

$g \longmapsto \max\{n : g \in \langle \pi^n \rangle\}$.

The valuation val_D on X is the divisorial valuation corresponding to Y.

$U \subseteq X$ open, $D = V(f)$, $K[U]_f$ DVR.

local with maximal ideal $\langle \pi \rangle$

Thus there is a valuation

$$\mathrm{val}_D : K(Y) \cong K(X) \to \mathbb{R} \cup \infty$$

$$g \longmapsto \max\{n : g \in \langle \pi^n \rangle\}.$$

The valuation val_D on X is the divisorial valuation corresponding to Y.

Defn Fix $X \subseteq (K^*)^n$. For a valuation val on $K(X)$, we define $[\mathrm{val}] \in \mathbb{R}^n$ by

$$[\mathrm{val}] = (\mathrm{val}(x_1), \ldots, \mathrm{val}(x_n)).$$

eg $X = V(x+y-1) \subseteq (K^\bullet)^2$ (secretly $m_{0,4}$)

$Y = V(x+y-z) \subseteq \mathbb{P}^2$ affine chart $z \neq 0$ $x = \frac{X}{Z}$

$D_1 = \{x=0\}$ $\left(\frac{K[x,y]}{x+y-1}\right)_x$ is a DVR

is $K[x]_x$

$\mathrm{val}_{D_1}(x) = 1$ $\mathrm{val}_{D_1}(y) = \mathrm{val}_{D_1}(1-x) = 0$

$[D_1] = (1,0)$.

$D_2 = \{y=0\}$ $\mathrm{val}_{D_2}(x) = 0$ $\mathrm{val}_{D_2}(y) = 1$

$[D_2] = (0,1)$

$X = V(x+y-1) \subseteq (K^*)^2 \qquad Y = V(x+y-z) \subseteq \mathbb{P}^2$

$D_3 = \{z=0\}$. Chart $x \neq 0$

$y = \frac{y}{x}$

$z = \frac{z}{x}$

$$\left(\frac{K[y,z]}{1+y-z} \right)_z$$

$\mathrm{val}\left(\frac{x}{z}\right) = -\mathrm{val}\left(\frac{z}{x}\right) = -1$

$\mathrm{val}\left(\frac{y}{z}\right) = \mathrm{val}\left(\frac{y}{x}\right) - \mathrm{val}\left(\frac{z}{x}\right)$
$= 0 - 1 = -1$

So $[D_3] = (-1,-1)$.

$D_1 : \{x=0\}$ $[D_1] = (1,0)$ $D_2 : \{y=0\}$ $[D_2] = (0,1)$

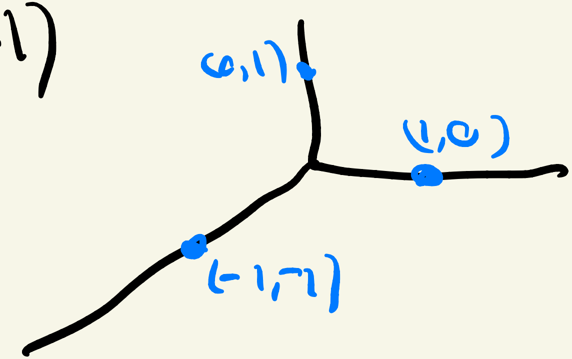
$D_3 : \{z=0\}$ $[D_3] = (-1,-1)$.

Theorem For $X \subseteq (K^*)^n$ (trivially valued)

$$\operatorname{trop}(X) = \operatorname{cl}\left(c[\mathrm{val}_D] : \mathrm{val}_D \text{ is a divisorial valuation on } X \right\}$$

$c \in \mathbb{Q}_{\geq 0}$

eg $X = V(x+y-1)$



Q How do we get all divisorial valuations?

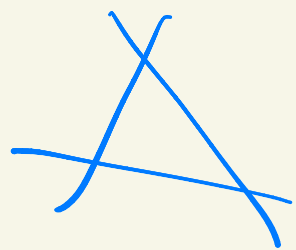
Fix smooth $X \subseteq (K^*)^n$. Let $\overline{X}$ be a smooth simple normal crossings compactification of X. This a projective variety with boundary $\overline{X} \setminus X$ equal to a union $D_1 \cup \cdots \cup D_s$ of divisors. The snc condition means localy any intersection of divisors looks like the intersection of coordinate lines.

Fix smooth $X \subseteq (K^*)^n$. Let $\overline{X}$ be a smooth simple normal crossings compactification of X.

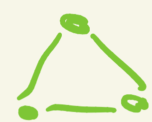
The boundary complex $\Delta(\overline{X}, X)$ is a simplicial complex with one vertex for each divisor $D_1, \dots, D_s$ & a simplex $\sigma \subseteq \{1, \dots, s\}$ whenever $\bigcap_{i \in \sigma} D_i \neq \emptyset$

eg $X = (\mathbb{C}^*)^2$

$\overline{X} = \mathbb{P}^2$

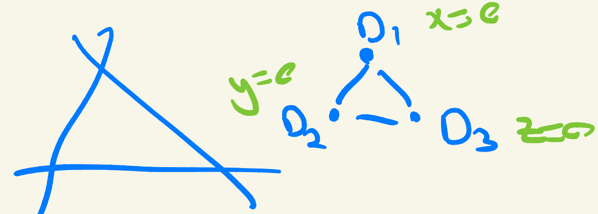


Δ:



Theorem With these hypotheses, $\mathrm{trop}(X)$ is the cone over the embedding of $\Delta(\overline{X}, X]$ into $\mathbb{R}^n$ that sends the vertex of D_i to $[\mathrm{val}_{D_i}]$.

eg $X = (K^*)^2$ $\quad \overline{X} = \mathbb{P}^2$



$[\mathrm{val}_{D_1}] = (1,0)$

$[\mathrm{val}_{D_2}] = (0,1)$

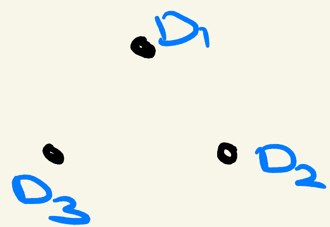
$[\mathrm{val}_{D_3}] = (-1,-1)$

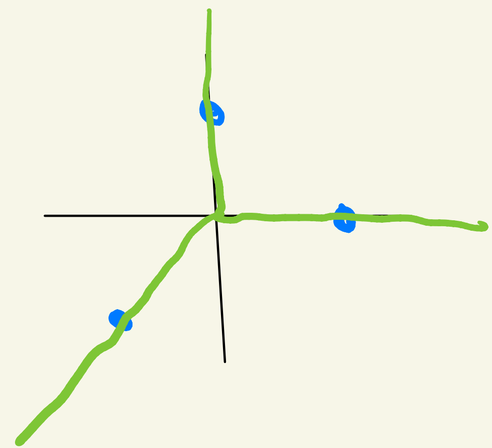
$\mathrm{trop}(K^*)^2 = \mathbb{R}^2$ as expected.

Theorem With these hypotheses, trop(X) is the cone over the embedding of $\Delta(\bar{X}, X]$ into $\mathbb{R}^n$ that sends the vertex of D_i to $[\mathrm{val}_{D_i}]$.

eg $X = V(x+y-1)$ $\quad \bar{X} = V(x+y-z)$

$\Delta(\bar{X}, X)$





References

Maclagan "Introduction to tropical algebraic geometry"

arXiv: 1207.1925

Maclagan-Sturmfels "Introduction to tropical geometry" AMS GSM

(for more details)

today was part of chapter 6