Introduction to Tropical Geometry III

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Last lecture:

Fundamental & Structure theorems

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Fundamental & Structure theorems

$\mathrm{trop}(X) = \overline{\mathrm{val}(X)}$.

Today: More details, and some useful tools

Structure theorem

$X \subseteq (K^*)^n$ irreducible $\Rightarrow$ $\mathrm{trop}(X)$ is the support of a weighted balanced polyhedral complex, such that . . . .

Q: Where do the weights come from?

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A/ Fix a facet P of Σ ← polyhedral complex structure on $\mathrm{trop}(X)$

& $w \in \mathrm{relint}(P)$.

interior of span (P).

Geometrically: We form X_w = limit of "flowing X in direction w". This is a union of $\mathrm{mult}(w)$-torus orbits $(K^*)^d$ counted with multiplicity

Algebraically:

Choose a splitting $\Gamma = \operatorname{im} \mathrm{val} \to K$

$\omega \mapsto t^{\omega}$

$\mathrm{val}(t^{\omega}) = \omega.$

Set $R = \{ a \in K : \mathrm{val}(a) \geq 0 \}.$ ← local ring

$m = \{ a \in K : \mathrm{val}(a) > 0 \}$

$\Bbbk = R/m$ ← residue field

for $a \in R$ $\bar{a}$ is image in $\Bbbk$

For $f = \sum c_u x^u \in I(X)$

$$\mathrm{in}_{\underline{w}}(f) = \sum_{\substack{\mathrm{val}(c_u) \\ + w\cdot u \ \min}} \overline{c_u t^{-\mathrm{val}(c_u)}}\, x^u \in \Bbbk[x_1^{\pm}, x_n^{\pm}]$$

eg $f = 2x^2 + 3xy + 4y^2 \in \mathbb{Q}[x^{\pm}, y^{\pm}]$ $\quad \mathrm{val}_2$

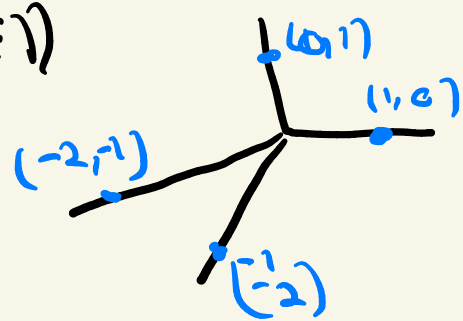
$\underline{w} = (1,2)$ $\quad \min(\mathrm{val}(c_u) + w\cdot u) = 3$

$$\mathrm{in}_w(f) = \overline{2 \cdot 2^{-1}}\, x^2 + \overline{3}\, xy = x^2 + xy \in \mathbb{Z}/2\mathbb{Z}[x^{\pm}, y^{\pm}]$$

$$\mathrm{in}_w(I(X)) = \langle \mathrm{in}_w(f) : f \in I(X) \rangle \subseteq \Bbbk[x_1^{\pm}, , x_n^{\pm}]$$

$V(\mathrm{in}_w(I(X)))$ is a union of $\mathrm{mult}(w)$ torus orbits, counted with multiplicity eg $f = x^3 + x^2y^2 + y^3 + 1$

$\mathrm{trop}(V(f))$



$w=(1,0)$ $\quad in_w(f) = y^3+1 \quad V(y^3+1) =$ 3 lines

so multiplicity 3

$w=(-1,-2)$ $\quad in_w(f) = x^2y^2+y^3$

$V(x^2y^2+y^3) = V(x^2+y)$ ← one subtorus $(t,-t^2)$

multiplicity

Note! $in_w(I)$ is a generalization of initial ideals from Gröbner theory.

The polyhedral complex structure on $trop(X)$ comes from the existence of the Gröbner complex

finite polyhedral complex where $in_w(I) = in_{w'}(I)$ for w, w' in the same cell.

(generalizes Gröbner fan)

Changes of coordinates

Automorphisms of $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ are given by matrices $A \in GL_n(\mathbb{Z})$ and $\lambda \in (K^*)^n$:

$\varphi_A^* : x^{\underline{u}} \longmapsto x^{A\underline{u}}$

$\varphi_\lambda^* : x_i \longmapsto \lambda_i x_i$

eg $A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ $\quad x + y + 1 \longmapsto xy^2 + x^2y^3 + 1$

$\lambda = (4, 8)$ $\quad x + y + 1 \longmapsto 4x + 8y +$

Lemma Fix $\varphi^\bullet = \varphi^\bullet_\lambda \circ \varphi^\bullet_A : K[x_1^{\pm1}, \dots, x_n^{\pm1}] \to K[x_1^{\pm1}, \dots, x_n^{\pm1}]$, and $X = V(I) \subseteq (K^\bullet)^n$.

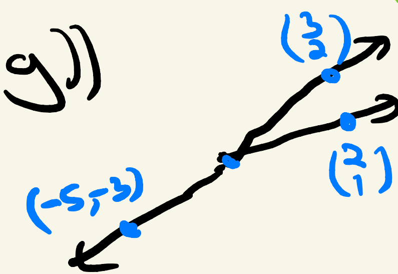
Then $\mathrm{trop}(\varphi^{\bullet-1}(I)) = A^T \mathrm{trop}(X) + \mathrm{val}(\lambda)$

eg $X = V(\overbrace{x+y+1}^{f})$, $A = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$

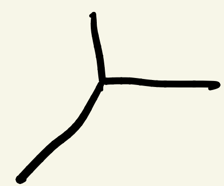
val_2: $\varphi^{\bullet-1}\langle f \rangle = \langle \overbrace{\frac{x^2}{y^3} + \frac{y^2}{x} + 1}^{g} \rangle$ (trick: use A^{-1})

check: $\frac{x^2}{y^3} \mapsto \frac{x^4}{y^6} \cdot \frac{y^6}{x^3} = x$

$\mathrm{trop}(V(g))$



$= \underbrace{\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}}_{A^T} \cdot$



Projections

Let $X \subseteq (K^\bullet)^n$ be a variety, and let $\pi: (K^\bullet)^n \longrightarrow (K^\bullet)^d$ be projection onto the first d coordinates (and also $\pi: \mathbb{R}^n \longrightarrow \mathbb{R}^d$)

Then $\operatorname{trop}(\pi(X)) = \pi(\operatorname{trop}(X))$

Pf Idea: Use fundamental theorem.

$$\pi(\operatorname{trop}(X)) = \operatorname{cl}\{(\operatorname{val}(x_1), \ldots, \operatorname{val}(x_d)) : x = (x_1, \ldots, x_n) \in X\}$$
$$= \operatorname{trop}(\pi(X))$$

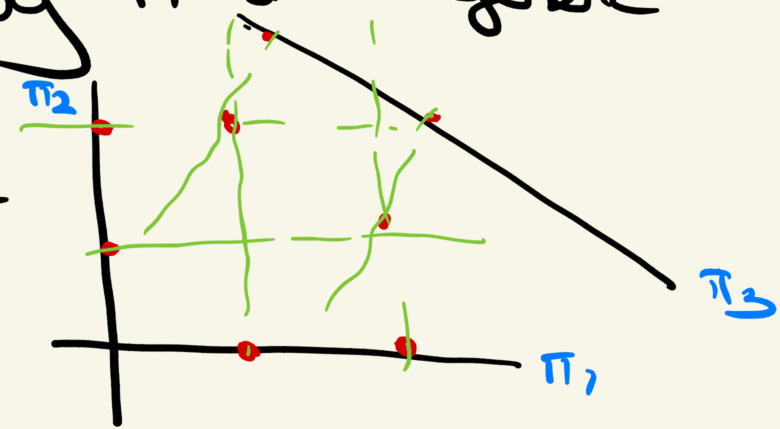
Get other projections by change of coordinates!

Corollary Tropical bases exist: Given $I \subseteq k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ there are $f_1, \ldots, f_s \in I$ with

$$\operatorname{trop}(V(I)) = \bigcap_{i=1}^{s} \operatorname{trop}(V(f_i))$$

Idea of pf: Project to hypersurfaces.

If P is a d-dim polyhedron in $\Sigma \subseteq \mathbb{R}^n$, it is determined by $n-d+1$ generic projections to $\mathbb{R}^{d+1}$



So choose $n-d+1$ generic projections to $\mathbb{R}^{d+1}$

These are hypersurfaces, corresp to $f_1, \ldots, f_{n-d+1}$

Connectedness

When X is irreducible, $\operatorname{trop}(X)$ is connected.

Idea of proof Induction on dimension.

Use **tropical Bertini theorem**: If X is irreducible, then for most hyperplanes H, $\operatorname{trop}(X) \cap H = \operatorname{trop}(Y)$ for some irreducible variety Y.

$\operatorname{trop}(Y)$ connected $\Rightarrow$ $\operatorname{trop}(X)$ connected

(choose H to pass through facets you want to connect)

That was the induction step. The base case is dim 1 (tropical curves).

This is the hard case!!!

Challenge: Find an elementary pf that the tropicalization of an irreducible curve is connected.

(Pfs of everything else I have told you can be understood with not much more than undergraduate background)

Proofs I know:

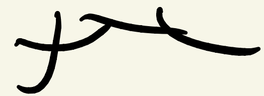
1. Since X is irreducible, the Berkovich analytification is connected. There is a cts map $X^{an} \to trop(X)$ so X^{an} connected $\Rightarrow trop(X)$ connected. (50 pages of Berkovich theory...)

(2) There is a complete Γ-rational polyhedral complex Σ containing $\mathrm{trop}(C)$ as a subcomplex. This defines a toric scheme over $\mathrm{Spec}(R)$, and so a family

support is $\mathbb{R}^n$

$$\begin{array}{c}\tilde{C}\\ \downarrow\\ \mathrm{Spec}(R)\end{array}$$

with general fiber C & special fiber a nodal curve C^0



with dual graph $\mathrm{trop}(C)$.

vertex for component, edge for intersection

By hard connectedness results, C^0 is connected, so $\mathrm{trop}(C)$ is connected.