Introduction to Tropical Geometry II

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Last lecture:

Tropical semiring $\overline{\mathbb{R}} = (\mathbb{R} \cup \infty, \underset{\min}{\oplus}, \underset{+}{\otimes})$

Tropical polynomials are piecewise linear fcns

Valuations: $\mathrm{val}(ab) = \mathrm{val}(a) + \mathrm{val}(b)$

$\mathrm{val}(a+b) \geq \min(\mathrm{val}(a), \mathrm{val}(b))$

$f = \sum a_u x^u \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ **trop(f)** $= \bigoplus \mathrm{val}(a_u) x^u$

$\in \overline{\mathbb{R}}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$

trop(V(f)) $= \{ w \in \mathbb{R}^n :$ min in $\mathrm{trop}(f)(w)$ is achieved $\geq 2 \}$

$I \subseteq K[x_1^{\pm}, \ldots, x_n^{\pm 1}]$ **trop(V(I))** $= \bigcap_{f \in I} \mathrm{trop}(V(f))$

Today: Fundamental and Structure theorems

Fix a field K. We'll need a valued field extension

This is L/K with valuation $\mathrm{val}_L : L \to \mathbb{R} \cup \infty$

with $\mathrm{val}_L(a) = \mathrm{val}(a)$ for $a \in K$.

Fundamental theorem

(Kapranov, Speyer-Sturmfels, Draisma, ...)

Fix a field K. Let $X = V(I) \subseteq (K^*)^n$ for "$(K \setminus \{0\})^n$"

$I \subseteq K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$

then $\mathrm{trop}(X) = \bigcap_{f \in I} \mathrm{trop}(V(f)) \subseteq \mathbb{R}^n$

euclidean top. $\longrightarrow$ $= \mathrm{cl}(\mathrm{val}(X_L) : L/K$ alg. closed nontrivially valued field extn$)$

$= \mathrm{cl}\{(\mathrm{val}(x_1), \ldots, \mathrm{val}(x_n)) : x = (x_1, \ldots, x_n) \in (L^*)^n$, $f(x) = 0 \ \forall f \in I$, L/K alg. closed nontrivially valued. $\}$

Generalizes val(root of f) = root of trop(f).

$\mathrm{trop}(X) = \mathrm{cl}(\mathrm{val}(X_L))$

eg $X = V(x+y-1) \subseteq (\mathbb{C}^*)^n$

$\mathrm{trop}(X) =$ 

$X_L = \{(a, 1-a) : a \in \cancel{\mathbb{C}}^{L^*} \setminus \{1\}\}$

$L/\mathbb{C}$ nontrivially field extn. eg $\mathbb{C}\{\{t\}\}$

- $\mathrm{val}(a) > 0$ $\mathrm{val}(1-a) = 0$
- $\mathrm{val}(a) < 0$ $\mathrm{val}(1-a) = \mathrm{val}(a)$
- $a = 1-b$ $\mathrm{val}(b) > 0$ $\mathrm{val}(1-a) = \mathrm{val}(b) > 0$

Fix a valued field K. Let $\Gamma = \operatorname{im} \operatorname{val} \subseteq \mathbb{R}$

Structure theorem Let $X \subseteq (K^*)^n$ be an irreducible variety of dim d. Then

$X \neq X_1 \cup X_2$

X_1, X_2 proper subvarieties

$\operatorname{trop}(X)$ is the support of a pure balanced Γ-rational polyhedral complex that is d-connected through codimension-one.

Rest of lecture Make sense of this!

Credits: Bieri-Groves, BJSST, Cartwright-Payne, M-Yu, CHHMMRWY, ...

Fix a valued field K. Let $\Gamma = \operatorname{im} \operatorname{val} \subseteq \mathbb{R}$

Structure theorem Let $X \subseteq (K^*)^n$ be an irreducible variety of dim d. Then $\operatorname{trop}(X)$ is the support of a pure balanced Γ-rational polyhedral complex that is d-connected through codimension-one.

$X \neq X_1 \cup X_2$

X_1, X_2 proper subvarieties

$\mathbb{Q}, \operatorname{val}_2 \quad \Gamma = \mathbb{Z}$

$\mathbb{C}\{\{t\}\} \quad \Gamma = \mathbb{Q}$

Defn A **polyhedron** $P \subseteq \mathbb{R}^n$ is a set of the form

$$\{x \in \mathbb{R}^n : Ax \le \underline{b}\} = \{\underline{x} \in \mathbb{R}^n : \underline{a}_1 \cdot \underline{x} \le b_1,\ \underline{a}_2 \cdot \underline{x} \le b_2,\ \vdots,\ \underline{a}_d \cdot \underline{x} \le b_d\}$$

A: $d \times n$ matrix

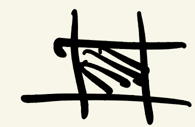
$\underline{b} \in \mathbb{R}^d$

"intersection of closed half spaces"

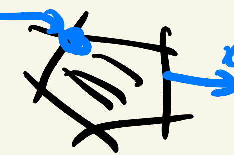
eg

$$\{\underline{x} \in \mathbb{R}^2 : \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \underline{x} \le \begin{pmatrix} 0 \\ 0 \end{pmatrix}\}$$

A **polytope** is a bounded polyhedron



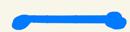
P is **Γ-rational** if $A \in \Gamma^{d \times n}$, $\underline{b} \in \Gamma^d$



Defn The linear span of P is
$\operatorname{span}(\underline{x}-\underline{y} : \underline{x},\underline{y} \in P)$.

eg span() = 

The dimension of P is the dimension of its linear span.

dim () = 1

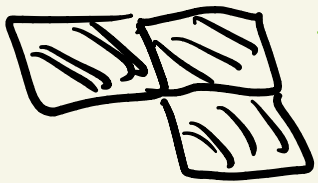
Defn The linear span of P is
$\text{span}(\underline{x}-\underline{y} : \underline{x}, \underline{y} \in P)$.

eg

The dimension of P is the dimension of its linear span.

Defn The face of P with (inner) normal vector $\underline{w}$ is
$$\text{face}_w(P) = \{x \in P : w \cdot x \leq w \cdot y \ \forall y \in P\}$$

Defn A **polyhedral complex** Σ in $\mathbb{R}^n$ is a finite collection of polyhedra, for which the intersection of any two is a face of each (or empty)



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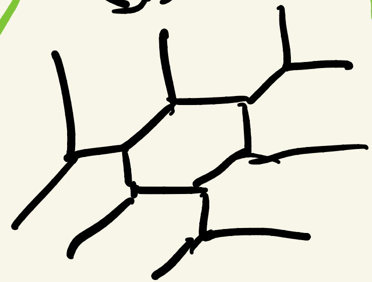

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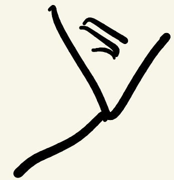

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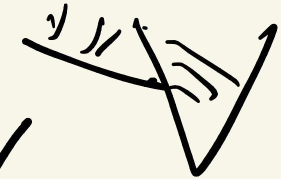

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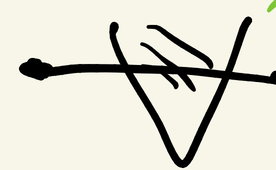

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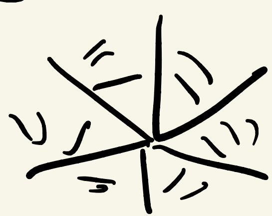
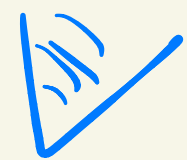
If all polyhedra are cones, Σ is a **fan**

$\{x : Ax \leq 0\}$





Fix a valued field K. Let $\Gamma = \operatorname{im} \operatorname{val} \subseteq \mathbb{R}$

Structure theorem Let $X \subseteq (K^*)^n$ be an irreducible variety of dim d. Then $\operatorname{trop}(X)$ is the support of a pure balanced Γ-rational polyhedral complex that is d-connected through codimension-one.

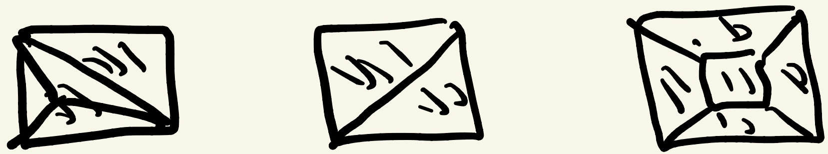
$X \neq X_1 \cup X_2$

X_1, X_2 proper subvarieties

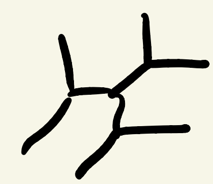
Defn The **support** of a polyhedral complex is the union of all polyhedra as a subset of $\mathbb{R}^n$.

Two polyhedral complexes can have the same support

eg



Defn A polyhedral complex Σ is **pure** if every maximal polyhedron in Σ (w.r.t inclusion) has the same dimension



"facet"

Fix a valued field K. Let $\Gamma = \operatorname{im} \operatorname{val} \subseteq \mathbb{R}$

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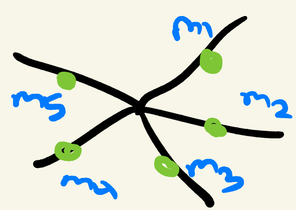
Balancing

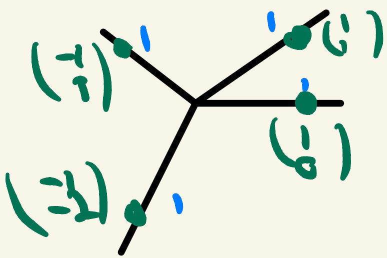
A **weighting** on a pure polyhedral complex is an assignment of an integers to each facet of Σ.

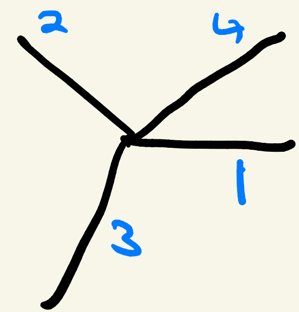
Defn A one-dimensional weighted rational polyhedral fan is **balanced** if $\sum m_i u_i = 0$ where m_i is the weight on the ith ray and u_i is the first lattice pt $\in \mathbb{Z}^n$ on the ith ray



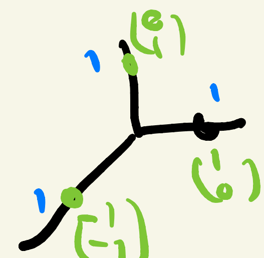
Defn A one-dimensional weighted rational polyhedral fan is **balanced** if $\sum m_i u_i = 0$ where m_i is the weight on the ith ray and u_i is the first lattice pt on the ith ray.

eg





$$1\binom{1}{0} + 4\binom{1}{1} + 2\binom{-1}{1} + 3\binom{-1}{-2} = \binom{0}{0}$$



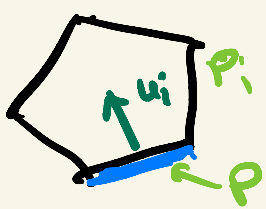
$$1\binom{1}{0} + 1\binom{0}{1} + 1\binom{-1}{-1} = \binom{0}{0}$$

Defn A pure d-dimensional Γ-rational polyhedral complex Σ is **balanced** if, for every $(d-1)$-dimensional polyhedron P in the complex, the following one-dimensional rational fan is balanced:

Let $P_1, \ , P_s$ be the d-dimensional polytopes in Σ containing P as a face. $\{u_{ij}\}$ rel. prime

Write $\underline{u}_i \in \mathbb{Z}^n$ for the primitive inner normal. Let Δ be the one-dimensional fan in $\mathbb{R}^n / \operatorname{span}(P)$ with rays $\dfrac{\operatorname{pos}(u_i) + \operatorname{span}(P)}{\operatorname{span}(P)}$ with weight $\operatorname{weight}(P_i)$.

$\{\lambda \underline{u}_i : \lambda \geq 0\}$



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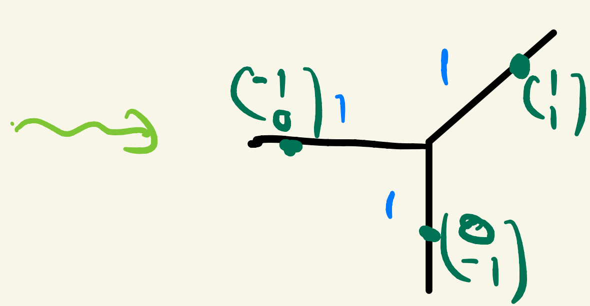
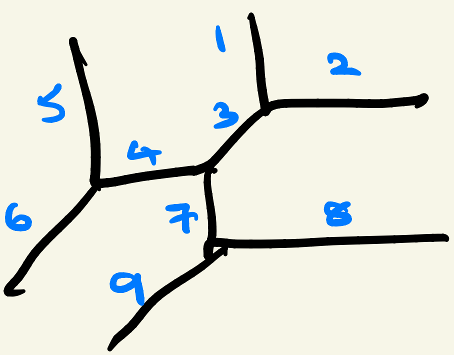


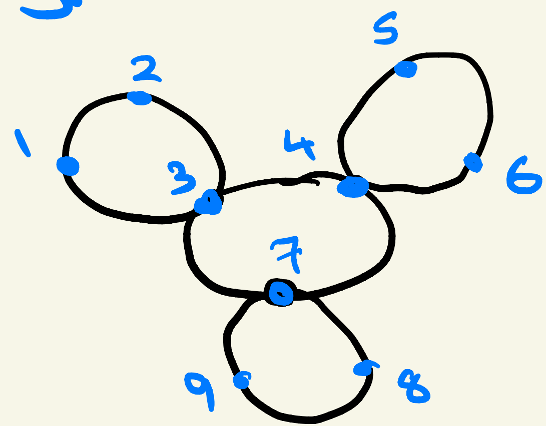
$$1\begin{pmatrix}-1\\0\end{pmatrix} + 1\begin{pmatrix}0\\-1\end{pmatrix} + 1\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

Defn A pure d-dimensional polyhedral complex is connected through codimension one if the facet-ridge hypergraph is connected

vertex for each d-dim polyhedron

hyperedge for each $(d-1)$-dim polyhedron

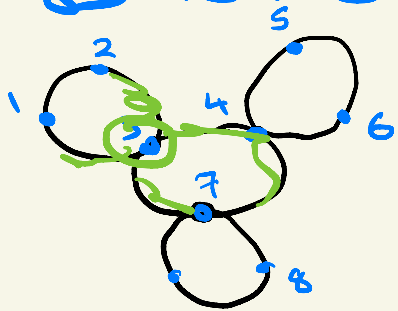




Defn A pure d-dimensional polyhedral complex is connected through codimension one if the facet-ridge hypergraph is connected.

It is d-connected through codimension one if this hypergraph is d-connected

(removing $(d-1)$ vertices & associated hyperedges leaves the hypergraph connected)



← connected = 1-connected but not 2-connected.

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X_1, X_2 proper subvarieties

$d - \dim(\text{lineality space } \Sigma)$

$\underline{u} \in \text{lineality space}(\Sigma)$

if $\underline{x} \in \Sigma \Rightarrow \underline{x} + \lambda \underline{u} \in \Sigma \quad \forall \lambda$