Introduction to Tropical Geometry

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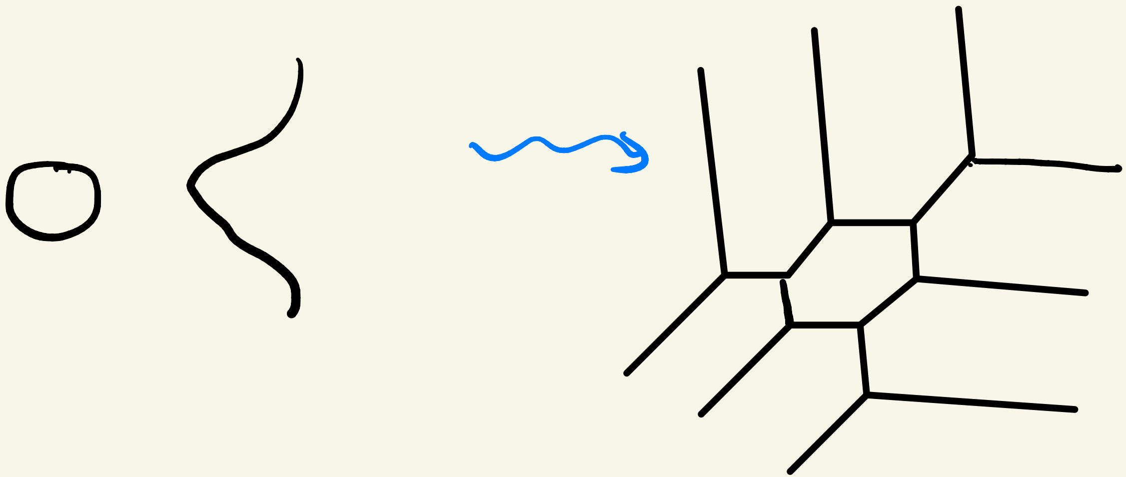
Slogan!: Tropical Geometry is a combinatorial shadow of algebraic geometry



Goal: Understand varieties via their tropicalizations

Slogan 2: Tropical geometry is algebraic geometry over the tropical semiring

$$\overline{\mathbb{R}} = (\mathbb{R} \cup \infty, \underset{\min}{\oplus}, \underset{+}{\otimes})$$

$3 \oplus 5 = 3$

$8 \otimes 7 = 15$

$6 \otimes (5 \oplus 3) = 9 = 6 \otimes 5 \oplus 6 \otimes 3$

$7 \oplus \infty = 7 \qquad 8 \otimes 0 = 8$

Fine print: Tropical geometry is really more than both these things: it also has applications outside mathematics

eg phylogenetics, economics,
machine learning, statistics, ...

Reference for today/tomorrow

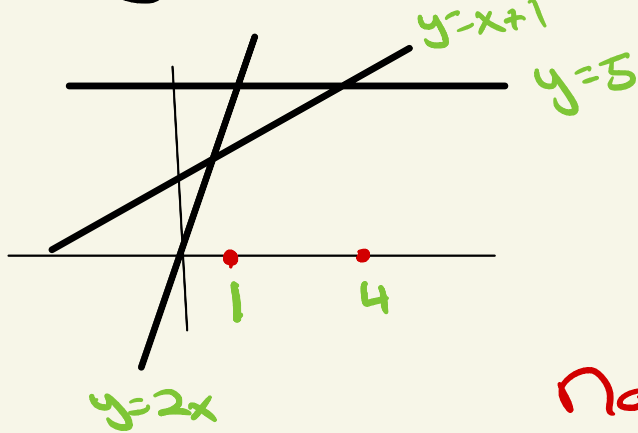
Maclagan "Introduction to tropical algebraic geometry"

arXiv: 1207.1925

Maclagan - Sturmfels "Introduction to tropical geometry" AMS GSM
(for more details)

Tropical polynomials are piecewise linear functions:

eg $x^2 \oplus 1 \otimes x \oplus 5 = \min(2x, x+1, 5)$



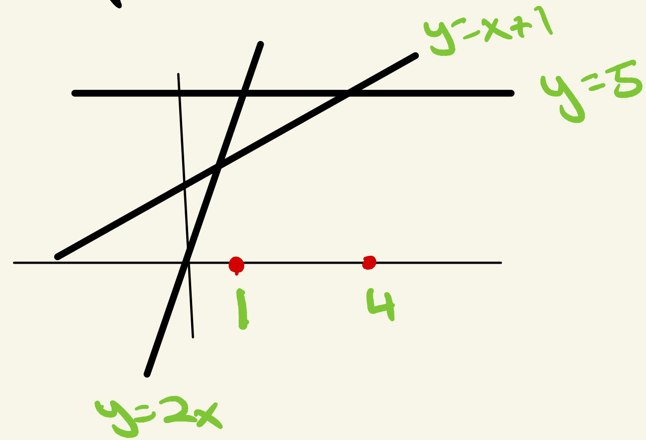
Note: $x^2 \oplus 1 \otimes x \oplus 5$

$= (x \otimes 1) \otimes (x \oplus 5)$

Q/ What are the roots of a tropical polynomial?

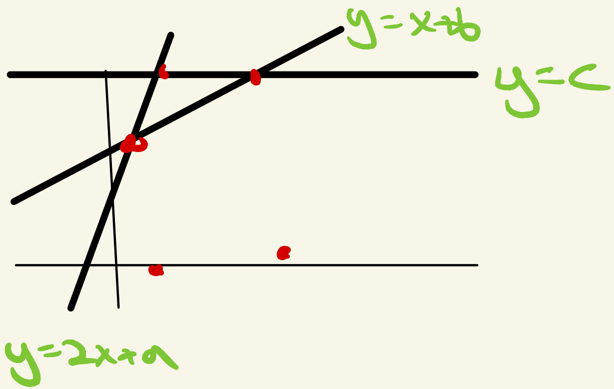
eg Can't solve $3 \oplus x = 5$

A/ **Defn** The roots of $f \in \mathbb{\overline{R}}[x]$ are the places where the graph is not differentiable so the min in the tropical sum is achieved at least twice.



Tropical quadratic formula:

$$f = a \otimes x^2 \oplus b \otimes x \oplus c = \min(2x+a, x+b, c)$$



Roots: $\begin{cases} c-b,\ b-a & a+c \geq 2b \\ \frac{c-a}{2} & a+c < 2b \end{cases}$

Challenge: Compute the tropical cubic formula

- - - quintic . . .

Connection to usual arithmetic

Let K be a field with a valuation

$val : K \to \mathbb{R} \cup \infty$ s.t.

$val(ab) = val(a) + val(b)$

$val(a+b) \geq \min(val(a), val(b))$

$val(a) = \infty$ if and only if $a = 0$

eg Any K, $val(a) = 0$ if $a \neq 0$

trivial valuation / constant coefficients

Connection to usual arithmetic

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eg $K = \mathbb{Q}$, $val = val_p$ p-adic valuation

$val_p(p^n \frac{a}{b}) = n \quad p \nmid a, b$

$val_2(6) = val_2(2 \times 3) = 1 \qquad val_2(\frac{5}{12}) = -2$

$val_3(9) = 2$

Connection to usual arithmetic

Let K be a field with a valuation

$\mathrm{val}: K \to \mathbb{R} \cup \infty$ s.t.

$\mathrm{val}(ab) = \mathrm{val}(a) + \mathrm{val}(b)$

$\mathrm{val}(a+b) \geq \min(\mathrm{val}(a), \mathrm{val}(b))$

$\mathrm{val}(a) = \infty$ if and only if $a = 0$

algebraically closed

eg $K = \mathbb{C}((t))$ ← Laurent series

$\mathrm{val}(\sum a_i t^i) = \min\{i : a_i \neq 0\}$

$K = \mathbb{C}\{\{t\}\} = \bigcup_{n \geq 0} \mathbb{C}((t^{\frac{1}{n}}))$ ← Puiseux series

$7t^{-2} + 8t^{-\frac{1}{3}} + 9t^{\frac{2}{3}} + 7t^{\frac{10}{3}} + \dots$

Connection to usual arithmetic

Let K be a field with a valuation

$val: K \to \mathbb{R} \cup \infty$ s.t

$val(ab) = val(a) + val(b)$

$val(a+b) \geq \min(val(a), val(b))$

$val(a) = \infty$ if and only if $a = 0$

Ex: If $val(a) \neq val(b)$ then

$val(a+b) = \min(val(a), val(b))$

Defn The tropicalization of $f = \sum a_i x^i \in K[x]$ is $\mathrm{trop}(f) = \bigoplus \mathrm{val}(a_i) \otimes x^i \in \overline{\mathbb{R}}[x]$

Key fact: (K algebraically closed valued field)

$$\{\text{roots of } f\} \xrightarrow{\text{val}} \{\text{roots of trop}(f)\}$$

$f(a) = 0 \quad \Rightarrow \quad \mathrm{val}(a)$ is a root of $\mathrm{trop}(f)$.

$\exists$ a with $\mathrm{val}(a) = b$, $f(a) = 0 \quad \Leftarrow \quad b$ root of $\mathrm{trop}(f)$

More variables

$f \in K[x_1^{\pm 1} \; x_n^{\pm 1}]$ $\quad f = \sum_{u \in \mathbb{N}^n} a_u x^u$ $\quad x_1^{u_1} x_2^{u_2} \cdot x_n^{u_n}$

$\mathrm{trop}(f) = \bigoplus \mathrm{val}(a_u) x^u$

$= \min(\mathrm{val}(a_u) + x \cdot u)$

piecewise linear fcn!

The tropical hypersurface of

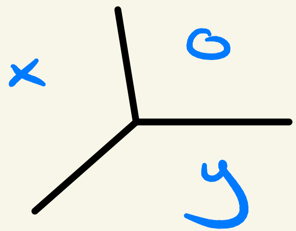
$f \in K[x_1^{\pm 1}, \quad x_n^{\pm 1}]$ is

$\mathrm{trop}(V(f)) = \{ w \in \mathbb{R}^n :$ the min in $\mathrm{trop}(f)$ is achieved $\geq 2 \}$

The tropical hypersurface of $f \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is

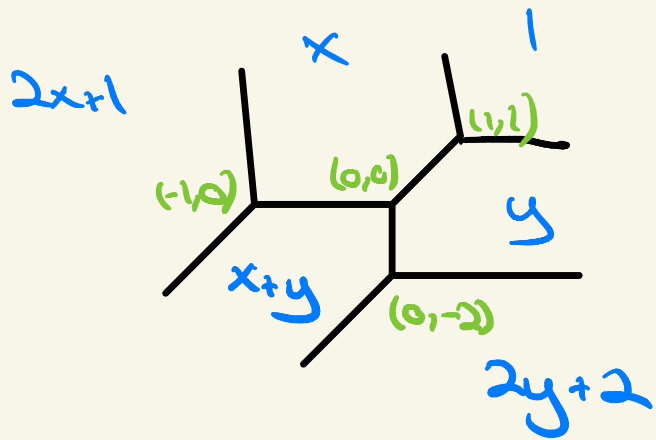
$$\operatorname{trop}(V(f)) = \{ w \in \mathbb{R}^n : \text{the min in } \operatorname{trop}(f) \text{ is achieved} \geq 2 \}$$

eg $f = x + y + 1$ $\qquad \operatorname{trop}(f) = x \oplus y \oplus 0$



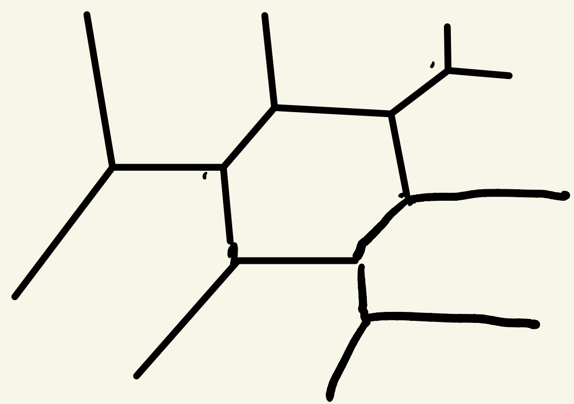
eg $f = 2x^2 + 3xy + 4y^2 + 7x + 9y + 6 \in \mathbb{Q}[x]$ $\quad \mathrm{val} = \mathrm{val}_2$

$\mathrm{trop}(f) = 1 \otimes x^2 \oplus xy \oplus 2 \otimes y^2 \oplus x \oplus y \oplus 1$



eg $f = 8x^3 + 2x^2y + 2xy^2 + 8y^3 + 2x^2 + xy$
$+ 2y^2 + 2x + 2y + 8$

$trop(f) = 3x^3 \oplus 1x^2y \oplus 1xy^2 \oplus 3y^3 \oplus 1x^2 \oplus xy \oplus 1y^2 \oplus 1x \oplus 1y \oplus 3$



Defn For an ideal $I \subseteq K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$

$$V(I) = \{ x \in (K^*)^n : f(x) = 0 \ \forall f \in I \}$$

$$\operatorname{trop}(V(I)) = \bigcap_{f \in I} \operatorname{trop}(V(f))$$