VALLADOLID ADDITIONAL EXERCISES

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This problem sheet is not designed to be done in order. You may want to do the first exercise from every section before returning to the rest, or jump straight to Question 12. There are also far too many questions to do during the time you have this week!

The tropical semiring. In the lecture we met $\overline{\mathbb{R}} = (\mathbb{R} \cup \{\infty\}, \oplus, \odot)$, where $\oplus$ is min, and $\odot$ is usual addition.

(1) Verify that this is a semiring (i.e., satisfies all the axioms of a ring, except that addition is an abelian monoid, not an abelian group).

There are many equivalent versions of the tropical semiring.

(2) Let $S_1 = (\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$, where $\odot$ is still addition, but $\oplus$ is now max. Verify that S_1 is a semiring, and that it is isomorphic to $\overline{\mathbb{R}}$. This is the "max convention", and is used by about half of tropical geometers.

(3) Let $S_2 = (\mathbb{R}_{\geq 0}, \oplus, \odot)$, where $\oplus$ is max, and $\odot$ is usual multiplication. Verify that S_1 is a semiring, and that it is isomorphic to $\overline{\mathbb{R}}$.

Valuations.

(4) Check the following consequences of the valuation axioms:
   (a) $\operatorname{val}(1) = 0$;
   (b) $\operatorname{val}(-a) = \operatorname{val}(a)$;
   (c) $\operatorname{val}(a+b) = \min(\operatorname{val}(a), \operatorname{val}(b))$ if $\operatorname{val}(a) \neq \operatorname{val}(b)$.

(5) Verify that the p-adic valuation is a valuation.

(6) Verify that the valuation on $\mathbb{Q}(t)$ given by $\operatorname{val}(\sum_{i=0}^{n} a_i t^i) = \min\{i : a_i \neq 0\}$ (the "low-degree" of the polynomial), and $\operatorname{val}(f/g) = \operatorname{val}(f) - \operatorname{val}(g)$, where $f, g \in \mathbb{Q}[t]$, is a valuation.

(7) The field of *Puiseux series* with complex coefficients is $\mathbb{C}\{\{t\}\} = \cup_{n \geq 1} \mathbb{C}((t^{1/n}))$, where $\mathbb{C}((t))$ is the field of Laurent series in t with complex coefficients. An entry of $\mathbb{C}\{\{t\}\}$ is a Laurent series in t with fractional exponents, where there is a least common denominator of the denominators appearing in the exponents. This has a valuation given by sending $a \in \mathbb{C}\{\{t\}\}$ to the smallest exponent on t appearing in a. Show that this is a valuation. What is the connection with the previous example? Bonus: Read a proof that $\mathbb{C}\{\{t\}\}$ is algebraically closed!

Polynomials of one variable.

(8) Show that every polynomial $f \in \overline{\mathbb{R}}[x]$ factors into a product of linear factors *as a function*. By this we mean that there is a polynomial $g = (x \oplus a_1) \odot \ldots \odot (x \oplus a_s) \in \overline{\mathbb{R}}[x]$ with $f(w) = g(w)$ for all $w \in \overline{\mathbb{R}}$. Give an example to show that f might not factor as a polynomial (so $f \neq g$ in general).

(9) What is the relationship between the a_i and the roots of f?

(10) Write down the tropical cubic formula.

(11) Bonus: Write down the tropical quintic formula! (or at least convince yourself that you could do it). Does this surprise you?

Tropical varieties.

(12) Draw $\operatorname{trop}(V(f))$ for the following $f \in \mathbb{Q}[x, y]]$, where $\mathbb{Q}$ has the 2-adic valuation.
  (a) $f = 8x + 6y + 5/4$;
  (b) $f = 3x + 4y + 48$;
  (c) $f = 8x^2 + xy + 10y^2 + 6x + y + 1$;
  (d) $f = 16x^2 + 6xy + 7y^2 + 7x + 5y + 2$;
  (e) $f = 2x^2 + 3xy - 7y^2 + 5$;
  (f) $f = 64x^3 + x^2y + xy^2 + 64y^3 + 8x^2 + 1/2xy + 8y^2 + 2x + 2y + 1$.

Plane tropical curves via regular triangulations. The goal of this exercise is to show the connection between tropical curves in the plane and triangulations of a certain point configuration (and give a short-cut to compute plane tropical curves!)

Fix $d > 0$. Let $\mathcal{A}_d = \{(a, b) : a + b \leq d, a, b \geq 0\}$. Fix a polynomial $f = \sum_{(a,b) \in \mathcal{A}_d} c_{ab} x^a y^b$ with $c_{ab} \in K$, where K is a field with a valuation. The *regular subdivision* of $\mathcal{A}_d$ induced by f is obtained by taking the convex hull of the points $\{(a, b, \operatorname{val}(c_{ab})) : (a, b) \in \mathcal{A}\}$ and taking the (projections of the) set of *lower faces*. These are the faces that you can see if you look from $(0, 0, -N)$ for $N \gg 0$.

Example: Let $d = 2$, so $\mathcal{A}_2 = \{(2, 0), (1, 1), (0, 2), (1, 0), (0, 1), (0, 0)\}$. Let $f = 2x^2 + xy + 6y^2 + x + y + 64$, where $\mathbb{Q}$ has the 2-adic valuation. We form the convex hull of the points

$$\{(2, 0, 1), (1, 1, 0), (0, 2, 1), (1, 0, 0), (0, 1, 0), (0, 0, 6)\}.$$

The lower faces of this polytope are illustrated in Figure 1.

(13) Draw the regular subdivision of $\mathcal{A}_2$ corresponding to the polynomial $f = 2x^2 + xy + 24y^2 + x + 10y + 1$.

(14) Draw the regular subdivision of $\mathcal{A}_1$ corresponding to the polynomial $f = 32x + 8y + 1024$.

(15) Draw the regular subdivision of $\mathcal{A}_3$ corresponding to the polynomial $f = 8x^3 + 2x^2y - 2xy^2 + 8y^3 + 10x^2 + xy + 6y^2 - 2x + 2y + 8$.

The dual graph to a subdivision has a vertex for every polygon (which will be triangles in most cases). There are two types of edges. The finite edges join two adjacent polygons, and have direction orthogonal to the common edge of the polygons.

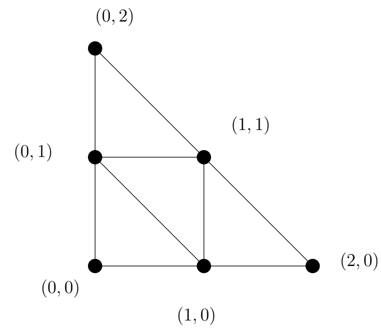

FIGURE 1.

The infinite edges start at the polygons adjacent to the boundary of the large triangle $\mathrm{conv}((d,0),(0,d),(0,0))$, and have direction orthogonal to the external edge. This is defined up to the lengths of the finite edges.

Example: In the example above, a dual graph for the regular subdivision (triangulation) is shown in Figure 2.

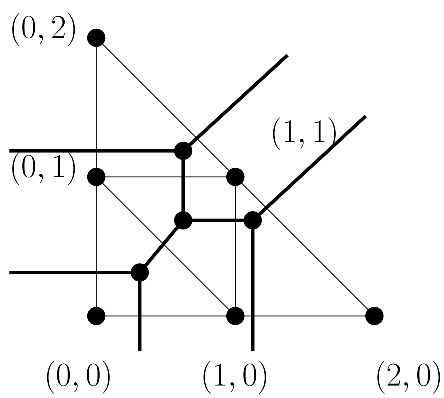

FIGURE 2.

(16) Draw a dual graph to the regular subdivision of $\mathcal{A}_2$ corresponding to $f = 2x^2 + xy + 24y^2 + x + 10y + 1$.
(17) Draw a dual graph to the regular subdivision of $\mathcal{A}_1$ corresponding to $f = 32x + 8y + 1024$.
(18) Draw a dual graph to the regular subdivision of $\mathcal{A}_3$ corresponding to $f = 8x^3 + 2x^2y - 2xy^2 + 8y^3 + 10x^2 + xy + 6y^2 - 2x + 2y + 8$.
(19) Let $f = \sum_{(a,b)\in\mathcal{A}_d} c_{ab}x^ay^b$ with $c_{d0}, c_{0d}, c_{00} \neq 0$. Show that the tropical curve defined by f is the image under $x \mapsto -x$ of a dual graph to the regular subdivision defined by f.
(20) Check the previous claim for the examples of the first question.

The fundamental theorem.

(21) Let $X = V(tx + (3t + t^2)y + 7) \subseteq \mathbb{A}^2_{\mathbb{C}\{\{t\}\}}$. Verify the fundamental theorem by first computing $\text{trop}(X)$, and then checking that this equals $\text{cl}(\text{val}(X))$.

(22) Prove the fundamental theorem for ideals in one variable.

The balancing condition.

(23) For each of the following polynomials in $\mathbb{C}[x, y]$, where $\mathbb{C}$ has the trivial valuation, compute $V(\text{trop}(f))$, including the multiplicities. Check that $V(\text{trop}(f))$ is balanced.
  (a) $f = 1 + x^3 + y^3$;
  (b) $f = x + y + x^2 + y^2 + x^3y + xy^3 + x^3y^2 + x^2y^3$;
  (c) $f = 1 + x^4y^2 + xy^5$.

(24) Consider $f = x + y + z + 1 \in \mathbb{C}[x, y, z]$. The tropicalization $V(\text{trop}(f))$ is a fan with rays the span of $\{(1, 0, 0), (0, 1, 0), (0, 0, 1), (-1, -1, -1)\}$, and six two-dimensional cones spanned by any two of the cones. Check this, and check that the multiplicity of each maximal cone is 1. Check that this fan is balanced.

(25) Prove the balancing condition for curves in the plane with the trivial valuation (i.e., $\text{trop}(V(f))$ for $f \in K[x, y]$ where K has the trivial valuation). Hint: Use the first section. Show that the multiplicity of $\mathbf{w}$ is the lattice length of the corresponding edge of the Newton polytope of f.