INTERNATIONAL SCHOOL ON TROPICAL GEOMETRY
TROPICAL IDEALS

EXERCISE SHEET 4

FELIPE RINCÓN

Exercise 1. Let $I \subseteq \overline{\mathbb{R}}[x,y,z]$ be the homogenization of the non-realizable tropical ideal discussed in the lecture (and in Exercise 4 on Day 3). Namely, I is generated by polynomials of the form $f = \bigoplus_{\mathbf{x}^{\mathbf{u}} \in C} \mathbf{x}^{\mathbf{u}}$, where C consists of exactly $k+2$ monomials in a 'standard triangle' of size k, and C is minimal with this property. (A 'standard triangle' of size k in $\overline{\mathbb{R}}[x,y,z]$ is a collection of monomials of the form

$$\Delta = \{x^{n+i} y^{m+j} z^{l+k-i-j} : i,j \geq 0 \text{ and } i+j \leq k\},$$

for fixed $k,l,n,m \in \mathbb{N}$.)

(1) Determine the Hilbert function of I.
(2) Compute the initial ideals $\mathrm{in}_{(1,1,0)}(I)$ and $\mathrm{in}_{(0,1,0)}(I)$ in $\mathbb{B}[x,y,z]$.
(3) Expand on the previous item by computing the whole Gröbner complex of I in $\mathbb{R}^3$, and determining the initial ideal corresponding to each cone.

Exercise 2. Use the weak Nullstellensatz for tropical ideals to show that if I is a maximal tropical ideal of $\overline{\mathbb{R}}[x_1,\ldots,x_n]$ and I contains no monomials then $I = I_{\mathbf{v}}$ for some $\mathbf{v} \in \overline{\mathbb{R}}^n$, where $I_{\mathbf{v}} := \{f \in \overline{\mathbb{R}}[x_1,\ldots,x_n] : \mathbf{v} \in V(f)\}$ (see Exercise 3 on Day 3).

Additional exercises:

Exercise 3. A tropical ideal $I \subseteq \overline{\mathbb{R}}[x_1,\ldots,x_n]$ is **irreducible** if $I = I_1 \cap I_2$ implies $I = I_1$ or $I = I_2$. Use the ascending chain condition to show that every tropical ideal can be decomposed as a finite intersection of irreducible tropical ideals.

Research problem: Can you give interesting examples of irreducible tropical ideals? Can you find an interesting condition that implies irreducibility? What can you say about the varieties of irreducible tropical ideals?

Exercise 4. Give an example of two tropical ideals I_1, I_2 such that neither $I_1 + I_2$ nor $I_1 \cap I_2$ are tropical ideals.

Research problem: Can you define interesting notions of sum and intersection for tropical ideals? Desirable properties would be, for instance, that $V(I_1 \cap I_2) = V(I_1) \cup V(I_2)$, and $V(I_1 + I_2) = V(I_1) \cap_{\text{st}} V(I_2)$.

Exercise 5. Say that a tropical ideal $I \subseteq \overline{\mathbb{R}}[x_1, \dots, x_n]$ is **naively prime** if $f \odot g \in I$ implies $f \in I$ or $g \in I$.

(a) Show that if $\mathbf{v} \in \mathbb{R}^n$ then the tropical ideal $I_{\mathbf{v}} := \{f \in \overline{\mathbb{R}}[x_1, \dots, x_n] : \mathbf{v} \in V(f)\}$ is naively prime (see also Exercise 3 on Day 3).

(b) Show that the ideal $I \subseteq \overline{\mathbb{R}}[x, y]$ defined as

$$I = \left\{ \bigoplus_{i=0}^{n} f_i(x) \odot y^i : \text{for all } i,\, f_i(x) \in \overline{\mathbb{R}}[x] \text{ satisfies } 0 \in V(f_i) \right\}$$

is a tropical ideal, but it is not naively prime.

Research problem: Can you find a useful notion of primality for tropical ideals (or their associated bend congruences / coordinate semirings)? For example, we probably want the trivial ideal $I = \{\infty\}$ to be prime, and if J is a linear ideal then $\text{trop}(J)$ should be prime as well. (The ideals discussed in parts (a) and (b) of this exercise are all of this form.) A 'good' notion of primality should probably imply that the corresponding variety is irreducible.