INTERNATIONAL SCHOOL ON TROPICAL GEOMETRY
TROPICAL IDEALS

EXERCISE SHEET 3

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Exercise 1. Consider the ideal $J = \langle x^2 - 1 \rangle \subseteq \mathbb{C}[x]$, where $\mathbb{C}$ is the field of complex numbers with the trivial valuation, and let $I = \operatorname{trop}(J) \subseteq \overline{\mathbb{R}}[x]$.

(1) Describe all polynomials in J of minimal support (with respect to inclusion).
(2) Find a (potentially infinite) set of tropical polynomials that generates I.
(3) What is the tropical variety of I?
(4) Can I be finitely generated?
(5) Write down a collection of equivalences $f \sim g$ that generates the congruence $\operatorname{bend}(I)$ on $\overline{\mathbb{R}}[x]$. Can you make this collection finite?
(6) Find a simple description of the coordinate semiring $\overline{\mathbb{R}}[x]/\operatorname{bend}(I)$. What 'dimension' would you say it has?

Exercise 2. Consider the ideals

$$J_1 = \langle (x + y + 1)(xy + x + y) \rangle \quad \text{and} \quad J_2 = \langle (x + y)(x + 1)(y + 1) \rangle$$

in $\mathbb{C}[x, y]$, where $\mathbb{C}$ has the trivial valuation. Let $I_1 = \operatorname{trop}(J_1)$ and $I_2 = \operatorname{trop}(J_2)$.

(1) Show that $V(I_1) = V(I_2)$.
(2) Show that even though I_1 and I_2 contain the same tropical polynomials of degree at most 3, they contain different tropical polynomials of degree 4, and thus $I_1 \neq I_2$.

Additional exercises:

Exercise 3. Let $\mathbf{v} \in \mathbb{R}^n$ and consider the ideal $I_{\mathbf{v}} = \{f \in \overline{\mathbb{R}}[x_1, \dots, x_n] : \mathbf{v} \in V(f)\}$. Show that $I_{\mathbf{v}}$ is generated by binomials, and use this to show that $I_{\mathbf{v}}$ is a tropical ideal.

Exercise 4. A 'standard triangle' of size k in the polynomial semiring $\overline{\mathbb{R}}[x, y]$ is a collection of monomials of the form

$$\Delta = \{x^{n+i} y^{m+j} : i, j \geq 0 \text{ and } i + j \leq k\},$$

for fixed $k, n, m \in \mathbb{N}$. Consider the ideal $I \subseteq \overline{\mathbb{R}}[x, y]$ generated by polynomials of the form $f = \bigoplus_{\mathbf{x^u} \in C} \mathbf{x^u}$, where C consists of exactly $k+2$ monomials in a 'standard triangle' of size k, and C is minimal with this property.

(1) Show the given generators of I satisfy the monomial elimination axiom, and conclude that I is a tropical ideal.

(2) Fill all the details in the following proof that I is a non-realizable tropical ideal.

Suppose $I = \mathrm{trop}(J)$ for some $J \subseteq K[x, y]$. We have that J contains a polynomial of the form $ax+by+c$. After scaling the variables, we can assume $a = b = c = 1$, so $x + y + 1 \in J$. It follows that $1 + x^3 + y^3 - 3xy \in J$. This leads to a contradiction.

(3) Show that the variety of I is equal to the standard tropical line $V(x \oplus y \oplus 0)$.