INTERNATIONAL SCHOOL ON TROPICAL GEOMETRY
TROPICAL IDEALS

EXERCISE SHEET 2

FELIPE RINCÓN

Exercise 1. Consider the linear subspace L of $\mathbb{C}\{\{t\}\}^4$ defined as

$$L = \text{rowspace} \begin{pmatrix} 1 & 0 & 1 & t \\ 0 & 1 & 1 & t^2 \end{pmatrix}.$$

(1) Compute generators for the ideal $I(L) \subset \mathbb{C}[x_1, \ldots, x_4]$ of all polynomial relations satisfied by L.
(2) Compute the (valuated) circuits of the valuated matroid $\mathcal{M}(L)$ on the ground set $\{1, \ldots, 4\}$.
(3) Write down a collection of tropical equations that describe the polyhedral complex $\text{trop}(L)$ in tropical projective space $\mathbb{R}^4 / \mathbb{R} \cdot \mathbf{1}$ (i.e., write down a tropical basis).
(4) Compute all the faces of the polyhedral complex $\text{trop}(L)$ explicitly.

Additional exercises:

Exercise 2. A subset $X \subset \overline{\mathbb{R}}^n$ is called **tropically convex** if for any $x_1, x_2 \in X$ and $\lambda_1, \lambda_2 \in \overline{\mathbb{R}}$ we have $\lambda_1 \odot x_1 \oplus \lambda_2 \odot x_2 \in X$. Two points $x, y \in \overline{\mathbb{R}}^n$ are called **tropically orthogonal**, denoted $x \perp y$, if $\min(x_1 + y_1, \ldots, x_n + y_n)$ is attained at least twice.

(1) Show that an intersection of tropically convex sets is a tropically convex set.
(2) Show that for any $a \in \overline{\mathbb{R}}^n$, the set

$$a^\perp := \{x \in \overline{\mathbb{R}}^n : x \perp a\}$$

is tropically convex.
(3) Conclude that any tropical linear space is tropically convex.
(4) For a set $A \subset \mathbb{R}^n$, define its **orthogonal set** to be

$$A^\perp := \{x \in \mathbb{R}^n : x \perp a \text{ for all } a \in A\}.$$

Show that a balanced polyhedral complex $L \subset \mathbb{R}^n$ is a tropical linear space if and only if $(L^\perp)^\perp = L$.