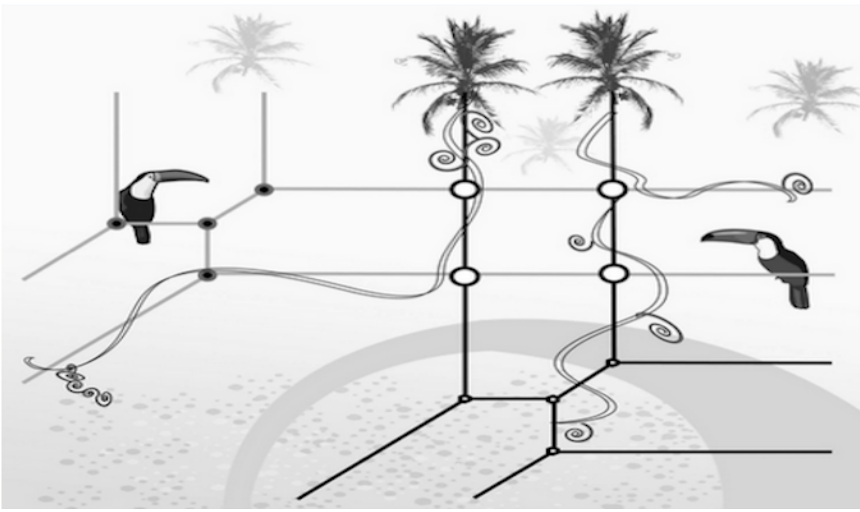
Counting Tropical Plane Curves

Renzo Cavalieri
following a master class by Hannah Markwig at Stockholm University

Notes for Students



(image by Cowdery and Challas, featured in June 2009 Mathematics Magazine)

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Introduction

The main goal of this minicourse is to introduce students to tropical curve counting techniques and their applications. Tropical geometry is a combinatorialization of ordinary geometry. By either doing algebraic geometry over the *tropical semi-field* or over *fields with a valuation*, one may associate piecewise linear objects to ordinary algebraic varieties. Remarkably such highly degenerate objects retain a lot of geometric information about the original algebraic varieties. For example, the notion of dimension is preserved: thus to an algebraic curve is associated a graph. Further, the genus of the curve generically is equal to the genus of the graph. In the last twenty-some years many groups of researchers have explored these connections, providing a wealth of *correspondence theorems*: statements on how to recover classical information from its tropical counterpart. In this mini-course we will focus in particular on a celebrated result of Mikhalkin, which shows that the count of rational tropical plane curves of degree d through $3d-1$ points in general position equals the classical count. The material will be distributed among the four classes as follows:

- **Lecture 1:** Cones, fans, and their morphisms. The notion of a balanced fan, and criteria to verify that a fan is balanced. The push-forward of a balanced fan is balanced.
- **Lecture 2:** Moduli spaces of tropical rational curves $M_{0,n}^{trop}$ as balanced fans. The forgetful morphisms.
- **Lecture 3:** Moduli spaces of tropical rational stable maps to the tropical plane. Evaluation morphisms and incidence conditions. The key enumerative question (*how many plane curves of a given degree pass through a number of points in general position?*), and how to interpret it in terms of a question of intersection on a moduli space of tropical stable maps.
- **Lecture 4:** Mikhalkin's correspondence theorem, recovering Kontsevich's recursive formula to answer the key enumerative question.

The material will be covered partly through lectures, partly through direct involvement of the students, whom will be asked to work on exercises aimed at helping them assimilate the material.

Acknowledgements

These notes are heavily based on a mini-course taught by Hannah Markwig during a master class at Stockholm University in the summer of 2017. Most of the credit for the choice and organization of the material goes to Hannah, and all of the blame for eventual mistakes rests on me.

I am very grateful to Andy Fry (Pacific University/Whitman College) for serving as TA for this activity.

References and Resources

There are many introductory resources to tropical geometry. Here is just a very incomplete selection for students that may be interested in reading more.

(1) *Introduction to Tropical Geometry.* Bernd Sturmfels and Diane Maclagan. Graduate Studies in Mathematics, AMS. Freely available online.
(2) *Brief introduction to tropical geometry.* Erwan Brugallè, Ilia Itenberg, Grigory Mikhalkin, Kristin Shaw. arXiv:1502.05950.
(3) *Enumerative tropical algebraic geometry in* $\mathbb{R}^2$ G. Mikhalkin. J. Amer. Math. Soc., 18(2):313 - 377, 2005.
(4) *Moduli spaces of rational tropical curves.* G. Mikhalkin. Proceedings of Gokova Geometry/Topology conference 2006, pages 39 - 51, 2007.
(5) *Fock Spaces and Tropical Curve Counting.* Renzo Cavalieri. https://www.math.colostate.edu//~renzo/CF.pdf

Part 1

Lectures

LECTURE 1

Cones and Fans

In this lecture we define the notions of cone (the intersection of a finite number of half spaces), and fan, which is a collection of cones assembled in a natural way. A fan is balanced if when you "hang" it by a codimension one cone τ and pull with certain amount of force along the incident top dimensional cones, it is only allowed to slide in the τ directions. We develop a more formal definition of balancing, as well as a criterion to easily check when a fan is balanced. Finally, we develop the notion of map of fans and push-forward fan, with the desirable property that pushing forward a balanced fan yields a balanced fan.

DEFINITION 1.1. A **rational polyhedral cone** $\sigma \subseteq \mathbb{R}^N$ can be defined in two equivalent ways:

(1) The non-negative span of a collection of vectors with rational coordinates:
$$\sigma = \left\{ \sum_{i=1}^{k} \lambda_i \mathbf{v_i} | \lambda_i \in \mathbb{R}^{\geq 0}, \mathbf{v_i} \in \mathbb{Q}^N \right\}. \tag{1}$$
(2) The intersection of a finite number of linear, rational, closed half-spaces:
$$\sigma = \bigcap_{i=1}^{l} H_i^+, \tag{2}$$
with
$$H^+ = \{\alpha_1 x_1 + \ldots + \alpha_N x_N \geq 0 | \alpha_i \in \mathbb{Q}\}. \tag{3}$$

A cone σ is called **strongly convex** if there isn't any non-zero vector $\mathbf{v}$ such that $\mathbf{v}$ and $-\mathbf{v}$ both belong to σ. If we take the first perspective and think of a cone as spanned by a set of vectors, given a strongly convex cone σ there is always a unique minimal set of primitive vectors generating σ. The half-lines inside the cone spanned by these vectors are called the **rays** of σ. A cone σ is called **simplicial** if it is strongly convex and its dimension equals the number of rays of σ.

Intuitively, a strongly convex cone is a cone which is "pointy". The rays of a cone are the one-dimensional "corners" of the cone.

DEFINITION 1.2. A **rational polyhedral fan** $\Sigma \subset \mathbb{R}^N$ is a collection of rational polyhedral cones with the property that any two cones intersect along faces:
$$\Sigma = \bigcup_{i=1}^{k} \sigma_i, \tag{4}$$

such that for any i, j, $\sigma_i \cap \sigma_j$ is a face of both σ_i and σ_j.

A **maximal cone** of Σ is any cone which is not a face of another cone of Σ. We say that a fan Σ is **pure dimensional** if all maximal cones have the same dimension. In this case, we call the dimension of maximal cones the **dimension** of Σ.

A **weight function** ω_Σ on a fan Σ is a function from the set of cones of Σ to the non-negative integers $\mathbb{Z}^{\geq 0}$.

Given a codimension one cone $\tau \in \Sigma$, we define a **normal vector to** τ **in** σ, denoted $\mathbf{u}_{\tau/\sigma}$, to be any vector in σ which descends to a generator of the lattice $\frac{Span(\sigma)\cap\mathbb{Z}^N}{Span(\tau)}$. Note that there are typically many choices for $\mathbf{u}_{\tau/\sigma} \in \mathbb{R}^N$; however, they all descend to the same vector in $\mathbb{R}^N/Span(\tau)$.

Definition 1.3. A pure dimensional, rational, polyhedral fan with a weight function is **balanced** if for every codimension one face τ of Σ we have:

$$\sum_{\sigma>\tau} \omega_\Sigma(\sigma)\mathbf{u}_{\tau/\sigma} = 0 \in \mathbb{R}^N/Span(\tau). \tag{5}$$

The notation $\sigma > \tau$ means: τ is a face of σ.

Definition 1.4. A **marking** on a rational, simplicial fan is a choice of an integral vector (not necessarily primitive) on each ray of the fan (see Figure 1.1). A fan with a marking is called a **marked fan**.

A marking on a fan $\Sigma \subset \mathbb{R}^N$ is a way to give the fan a weight function. Given any cone $\tau \in \Sigma$, let $\rho_1, \ldots, \rho_k$ be the rays bounding τ and $\mathbf{v_1}, \ldots, \mathbf{v_k}$ the corresponding vectors in the marking. We define:

$$\omega_\Sigma(\tau) := \left| \frac{Span(\tau) \cap \mathbb{Z}^N}{\mathbb{Z}\mathbf{v_1} + \ldots + \mathbb{Z}\mathbf{v_k}} \right|. \tag{6}$$

Equation (6) *has a geometric meaning very much related to the definition of intersection multiplicity of tropical curves from Exercise 5.4: the*

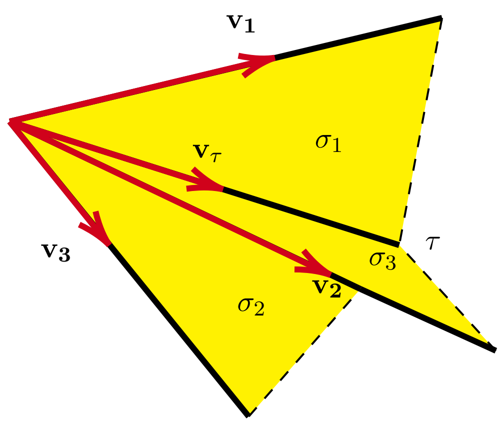


Figure 1.1. An illustration of a marking on a two dimensional fan.

weight of the cone σ *is the index (cardinality of the quotient) of the lattice generated by the markings inside the restriction of the ambient lattice to the subvector space spanned by the cone* σ. *As in* (57), *this can be computed as the absolute value of the determinant of a matrix expressing* $\mathbf{v_1}, \dots \mathbf{v_k}$ *as linear combinations of a minimal set of generators of the lattice* $Span(\tau) \cap \mathbb{Z}^N$.

A nice feature of weight functions induced by a marking is that one can easily check if a fan is balanced, as we show in the next Lemma.

LEMMA 1.1. *Consider a marked fan* Σ, *a codimension one cone* $\tau \in \Sigma$ *and a top dimensional cone* $\sigma > \tau$. *There is a unique vector in the marking that belongs to* σ *and does not belong to* τ, *denote it by* $\mathbf{v}_{\sigma \smallsetminus \tau}$. *Then* Σ *is a balanced fan if and only if*

$$\sum_{\sigma > \tau} \mathbf{v}_{\sigma \smallsetminus \tau} = 0 \in \mathbb{R}^N / Span(\tau). \tag{7}$$

PROOF. We rewrite the balancing condition (5) and show it is equivalent to (7). Let $\mathbb{R}^{K+1} \cong Span(\sigma) \supseteq \sigma > \tau \subseteq Span(\tau) \cong \mathbb{R}^K$. Up to the action of a matrix in $SL(K+1, \mathbb{Z})$, we may assume that τ is contained in the hyperplane $x_{K+1} = 0$. We then have the two following important facts:

$$\mathbf{v}_{\sigma \smallsetminus \tau} = x_{K+1}(\mathbf{v}_{\sigma \smallsetminus \tau}) \mathbf{u}_{\tau/\sigma} + \mathbf{t}_\sigma, \tag{8}$$

with $\mathbf{t}_\sigma \in Span(\tau)$.

$$\omega_\Sigma(\sigma) = x_{K+1}(\mathbf{v}_{\sigma \smallsetminus \tau}) \omega_\Sigma(\tau). \tag{9}$$

We now deduce

$$\sum_{\sigma > \tau} \omega_\Sigma(\sigma) \mathbf{u}_{\tau/\sigma} = \sum_{\sigma > \tau} x_{K+1}(\mathbf{v}_{\sigma \smallsetminus \tau}) \omega_\Sigma(\tau) \frac{\mathbf{v}_{\sigma \smallsetminus \tau} - \mathbf{t}_\sigma}{x_{K+1}(\mathbf{v}_{\sigma \smallsetminus \tau})} = \left(\omega_\Sigma(\tau) \sum_{\sigma > \tau} \mathbf{v}_{\sigma \smallsetminus \tau} \right) + \mathbf{t}, \tag{10}$$

with $\mathbf{t} \in Span(\tau)$. Since we assume Σ is a simplicial fan, $\omega_\Sigma(\tau) \neq 0$, which implies that

$$\sum_{\sigma > \tau} \omega_\Sigma(\sigma) \mathbf{u}_{\tau/\sigma} \in Span(\tau) \iff \sum_{\sigma > \tau} \mathbf{v}_{\sigma \smallsetminus \tau} \in Span(\tau). \tag{11}$$

$\square$

DEFINITION 1.5. Let $\Sigma_1 \subseteq \mathbb{R}^M$ and $\Sigma_2 \subseteq \mathbb{R}^N$. A map $f : |\Sigma_1| \to |\Sigma_2|$ is called a **map of fans** if f is the restriction of a $\mathbb{Z}$-linear map $\mathbb{R}^M \to \mathbb{R}^N$.

We would like a map of fans to be a function that sends cones to cones. Note that with this definition in place, this is not necessarily the case. However, given a map of fans, one may always subdivide some cones (both in the source and target fans) to obtain two new fans with the support, such that the same map now has the property of mapping cones to cones. So from now on when we talk about maps of fans let us assume that this additional property is verified.

DEFINITION 1.6. Given a map of fans $f : |\Sigma_1| \to |\Sigma_2|$, the **push-forward** of the fan Σ_1 via f is:

$$f_*(\Sigma_1) \coloneqq \left\{ f(\sigma) | \sigma \text{ is a maximal cone in } \Sigma_1 \text{ and } f_{|\sigma} \text{ is injective} \right\}. \tag{12}$$

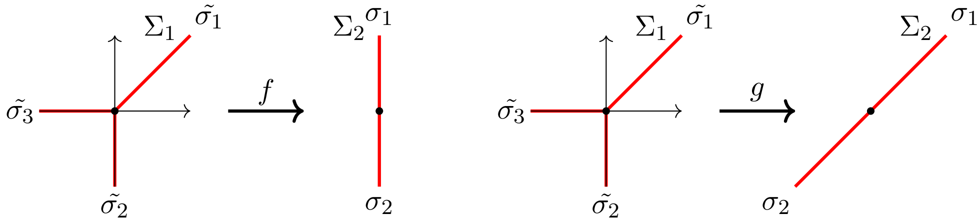


FIGURE 1.2. Two examples of maps of fans.

If Σ_1 has a weight function, we can induce a weight function on $f_*(\Sigma_1)$ as follows. If τ is a cone of $f_*(\Sigma_1)$, we define:

$$\omega_{f_*(\Sigma_1)}(\tau) := \sum_{\tilde{\tau} \ s.t. \ f(\tilde{\tau})=\tau} \omega_{\Sigma_1}(\tilde{\tau}) \left| \frac{Span(\tau) \cap \mathbb{Z}^N}{f(Span(\tilde{\tau}) \cap \mathbb{Z}^M)} \right|. \tag{13}$$

EXAMPLE 1.1. Consider the fans $\Sigma_1 \subseteq \mathbb{R}^2$, $\Sigma_2 \subseteq \mathbb{R}$ as depticted in Figure 1.2; all cones have weight one. In the first case, $f(x,y) = y$ is horizontal projection. Using (13), we have:

$$\omega_{f_*(\Sigma_1)}(\sigma_1) = \left| \frac{Span(\sigma_1) \cap \mathbb{Z}}{f(Span(\tilde{\sigma}_1) \cap \mathbb{Z}^2)} \right| = \left| \frac{\mathbb{Z} \cdot 1}{\mathbb{Z} \cdot f([1,1])} \right| = 1, \tag{14}$$

$$\omega_{f_*(\Sigma_1)}(\sigma_2) = \left| \frac{Span(\sigma_2) \cap \mathbb{Z}}{f(Span(\tilde{\sigma}_2) \cap \mathbb{Z}^2)} \right| = \left| \frac{\mathbb{Z} \cdot (-1)}{\mathbb{Z} \cdot f([0,-1])} \right| = 1. \tag{15}$$

We conclude that $f_*(\Sigma_1) = \Sigma_2$.

Now consider the function $g(x,y) = x + y$. Now we have

$$\omega_{g_*(\Sigma_1)}(\sigma_1) = \left| \frac{Span(\sigma_1) \cap \mathbb{Z}}{g(Span(\tilde{\sigma}_1) \cap \mathbb{Z}^2)} \right| = \left| \frac{\mathbb{Z} \cdot 1}{\mathbb{Z} \cdot g([1,1])} \right| = 2, \tag{16}$$

$$\begin{aligned} \omega_{g_*(\Sigma_1)}(\sigma_2) &= \left| \frac{Span(\sigma_2) \cap \mathbb{Z}}{g(Span(\tilde{\sigma}_2) \cap \mathbb{Z}^2)} \right| + \left| \frac{Span(\sigma_2) \cap \mathbb{Z}}{g(Span(\tilde{\sigma}_3) \cap \mathbb{Z}^2)} \right| \\ &= \left| \frac{\mathbb{Z} \cdot (-1)}{\mathbb{Z} \cdot g([0,-1])} \right| + \left| \frac{\mathbb{Z} \cdot (-1)}{\mathbb{Z} \cdot g([-1,0])} \right| = 1 + 1. \end{aligned} \tag{17}$$

We conclude that in this case $g_*(\Sigma_1) = 2\Sigma_2$.

LEMMA 1.2. *Let $\Sigma_1 \subseteq \mathbb{R}^M$ be a balanced fan, and f a map of fans. Then $f_*(\Sigma_1)$ is a balanced fan.*

DEFINITION 1.7. A balanced fan $\Sigma \subseteq \mathbb{R}^M$ is **irreducible** if it cannot be decomposed as the sum of two balanced fans with different supports. Here by sum of two balanced fans we mean taking the union of the cones in each fan, and adding the weights for any cone that appears in both fans.

EXAMPLE 1.2. Consider the two fans in Figure 1.3, where all cones have weight one. The fan Σ_1 is irreducible, while Σ_2 is not, as it may be decomposed as the sum of the two subfans $\Sigma_2' \cup \Sigma_2''$.

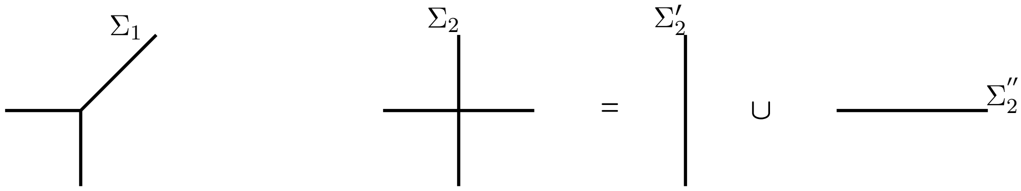


FIGURE 1.3. The fan Σ_1 is irreducible. The fan Σ_2 is reducible.

LEMMA 1.3. *Suppose Σ_1, Σ_2 are two balanced fans of the same dimension, Σ_1 is irreducible, and $|\Sigma_2| \subseteq |\Sigma_1|$. Then, up to subdivision, there exists a positive rational number λ such that*

$$\Sigma_1 = \lambda \Sigma_2. \tag{18}$$

We now come to the definition of the degree of a map of balanced fans.

DEFINITION 1.8. Let $\Sigma_1 \subseteq \mathbb{R}^M, \Sigma_2 \subseteq \mathbb{R}^N$ be two balanced fans of the same dimension, $f : \Sigma_1 \to \Sigma_2$ a map of fans.

$$mult_P(f) := \frac{\omega_{\Sigma_1}(\sigma_P)}{\omega_{\Sigma_2}(\sigma_{f(P)})} \left| \frac{Span(\sigma_{f(P)}) \cap \mathbb{Z}^N}{f(Span(\sigma_P) \cap \mathbb{Z}^M)} \right|. \tag{19}$$

Next assume Σ_2 is irreducible. We define the **degree** of f. For any point Q in the interior of a maximal cone of Σ_2:

$$deg(f) := \sum_{P \ s.t. \ f(P)=Q} mult_P(f). \tag{20}$$

The next theorem shows that the degree of f is well-defined.

THEOREM 1.1. *If Σ_2 is an irreducible fan, and $Q, \tilde{Q}$ are two points in the interior of maximal cones of Σ_2, then*

$$\sum_{P \ s.t. \ f(P)=Q} mult_P(f) = \sum_{P \ s.t. \ f(P)=\tilde{Q}} mult_P(f). \tag{21}$$

PROOF. Cosider the weighted fan $f_*(\Sigma_1)$. Since its support is contained in Σ_2 which is irreducible, by Lemma 1.3 there exists a rational number λ such that

$$\lambda \Sigma_2 = f_*(\Sigma_1). \tag{22}$$

This implies that for any point Q in the interior of a maximal cone of Σ_2,

$$\lambda \omega_{\Sigma_2}(\sigma_Q) = \omega_{f_*(\Sigma_1)}(\sigma_Q) = \sum_{P \ s.t. \ f(P)=Q} \omega_{\Sigma_1}(\sigma_P) \left| \frac{Span(\sigma_Q) \cap \mathbb{Z}^N}{f(Span(\sigma_P) \cap \mathbb{Z}^M)} \right|, \tag{23}$$

where the last equality is (13). The theorem is proved by dividing by $\omega_{\Sigma_2}(\sigma_Q)$ and observing that λ, which is independent of Q, equals the definition of $deg(f)$.

□

REMARK 1.1 (Very Important!). Assume Σ_1, Σ_2 are marked fans, denote $v_1, \dots, v_n$ the marking on the rays of a top dimensional cone $\sigma_P \in \Sigma_1$ and by $w_1, \dots, w_n$ the marking on the rays of $\sigma_Q = f(\sigma_P) \in \Sigma_2$. Then

$$mult_P(f) = det(M_f), \tag{24}$$

where M_f is the matrix representing the linear function f in the bases $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$.

To see that this is true, pick orthonormal bases β_1 for $Span(\sigma_P)$ and β_2 for $Span(\sigma_Q)$, and denote by $M^{\mathbf{v}}_{\beta_1}$ and $M^{\beta_2}_{\mathbf{w}}$ the matrices of the changes of bases with respect to the bases given by the markings. Also denote by $\widetilde{M}_f$ the matrix representing f in the two bases β_1 and β_2. Then we have:

$$M_f = M^{\beta_2}_{\mathbf{w}} \widetilde{M}_f M^{\mathbf{v}}_{\beta_1}, \tag{25}$$

which implies

$$det(M_f) = det(M^{\beta_2}_{\mathbf{w}}) det(\widetilde{M}_f) det(M^{\mathbf{v}}_{\beta_1}). \tag{26}$$

The remark now follows by observing that

$$det(M^{\beta_2}_{\mathbf{w}}) = \frac{1}{\omega_{\Sigma_2}(\sigma_Q)}, \quad det(\widetilde{M}_f) = \left| \frac{Span(\sigma_Q) \cap \mathbb{Z}^N}{f(Span(\sigma_P) \cap \mathbb{Z}^M)} \right|, \quad det(M^{\mathbf{v}}_{\beta_1}) = \omega_{\Sigma_1}(\sigma_P). \tag{27}$$

Example 1.3. Consider the cones $\sigma_1 = \langle v_1 = [1,0], v_2 = [0,1] \rangle$, and $\sigma_2 = \langle w_1 = [1,1], w_2 = [1,-1] \rangle$ both marked by the primitive vectors along their rays, and the linear function f mapping v_i to w_i. The matrix representing f using the vectors of the markings as basis vectors is the identity matrix, so according to the remark $mult_P(f) = 1$.

Using Definition 1.8, we obtain:

$$mult_P(f) := \frac{\omega_{\Sigma_1}(\sigma_P)}{\omega_{\Sigma_2}(\sigma_{f(P)})} \left| \frac{Span(\sigma_{f(P)}) \cap \mathbb{Z}^N}{f(Span(\sigma_P) \cap \mathbb{Z}^M)} \right| = \frac{1}{2} \left| \frac{\mathbb{Z}^2}{\langle w_1, w_2 \rangle_{\mathbb{Z}}} \right| = \frac{1}{2} \cdot 2 = 1. \tag{28}$$

LECTURE 2

Abstract tropical curves and their moduli

In this lecture we introduce the notion of an abstract tropical curve: this is a metric tree, where we do not worry about it living inside any particular ambient vector space. We construct a parameter space for the set of all trees with a fixed number of labeled ends; it naturally consists of a collection of cones glued along their faces. The punchline of today is that we can realize this space inside a very large vector space as a balanced fan, where all cones have weight 1. *This is called the moduli space of abstract, tropical,* n*-pointed, rational curves and is denoted* $M_{0,n}^{trop}$.

DEFINITION 2.1. An **abstract, rational, n-pointed tropical curve** is a tree T with n labeled ends, and a function (called the **metric**) from the set of edges $m : E(T) \to \mathbb{R}^{\geq 0}$. A curve is called **stable** if each vertex has valence ≥ 3.

We consider ends to be unbounded edges. In particular, there is only one vertex adjacent to an end. See Figure 2.1 for some examples.

If you start from the tropicalization of a rational curve (as in Diane's mini-course), which gives you a tropical curve in the plane, you get an abstract tropical curve by forgetting the embedding but remembering the length of all finite edges of the tropical curve. We allow the metric function to take value 0, *with the convention that we declare a graph with an edge of length zero equivalent to the graph obtained by contracting that edge.*

Forgetting the information of the metric for a tropical curve Γ, we obtain a stable tree, called the **topological type** of Γ.

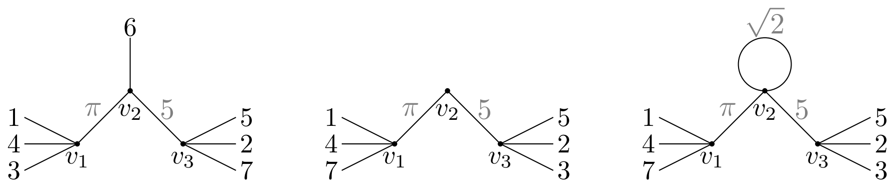


FIGURE 2.1. The first picture is an example of a stable, abstract, rational, tropical curve, with three vertices, two edges and seven labeled ends. The length of the edges is written in gray. The second picture is not stable because v_1 has valence 2. The third picture is not rational, as it has a loop starting and ending at v_2.

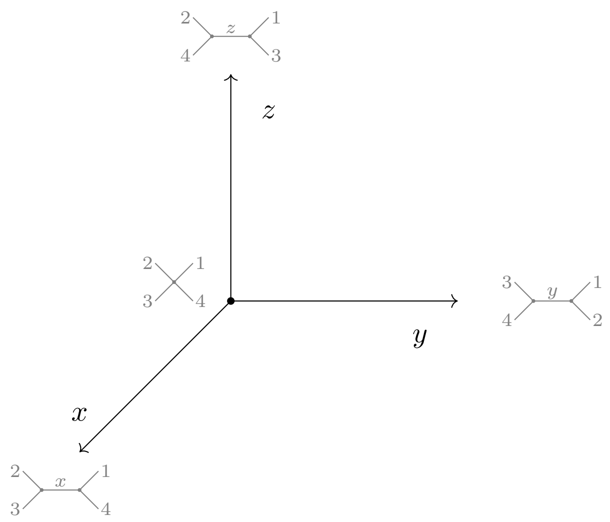


FIGURE 2.2. The cone complex $M_{0,4}^{trop}$ is obtained by identifying the vertices of three one dimensional cones (rays). The picture illustrates the cone complex, and in gray the tropical curves parameterized by each ray.

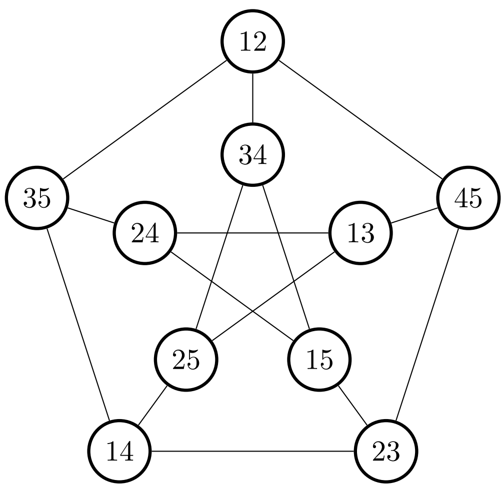


FIGURE 2.3. The cone complex $M_{0,5}^{trop}$ is the cone over the Petersen graph depicted here. Each ray parameterizes a tropical curve with one edge, two ends on one side, and three on the other. On the corresponding vertex of the Petersen graph we have marked the labels of the two ends that are together in the tropical curve.

Let T be a stable tree with n-ends and m edges. The set of all tropical curves of topological type T, or equivalently, the set of all possible metrizations of the edges of T, is naturally parameterized by the cone $(\mathbb{R}^{\geq 0})^m =: C_T$.

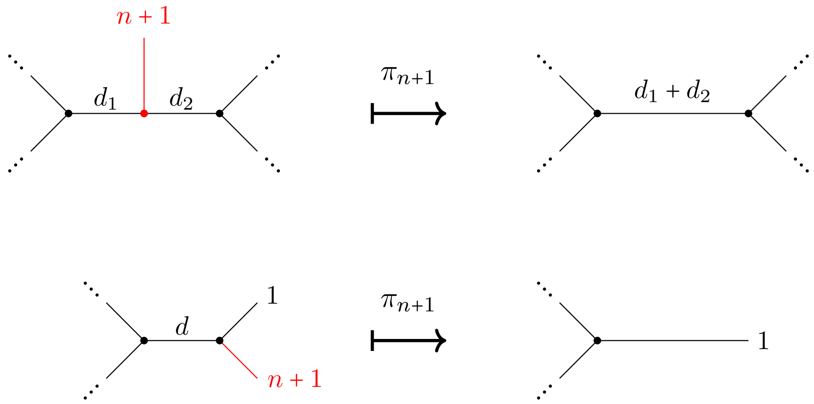


FIGURE 2.4. The process of stabilizing a graph after forgetting the $(n+1)$-th end.

DEFINITION 2.2. We denote by $M_{0,n}^{trop}$ the parameter space for stable, abstract, rational, n-pointed tropical curves. It consists of the cone complex

$$\coprod_T C_T \Big/ \sim, \tag{29}$$

where the disjoint union is over all topological types of stable trees with n ends, and two points $[\Gamma] \in C_T$, $[\Gamma'] \in C_{T'}$ are identified if $\Gamma \sim \Gamma'$, i.e. if they are equal after contracting all edges with length 0.

The space $M_{0,4}^{trop}$ is illustrated in Figure 2.2, the space $M_{0,5}^{trop}$ in Figure 2.3.

The **tropical forgetful morphim**

$$\pi_{n+1}^{trop} : M_{0,n+1}^{trop} \to M_{0,n}^{trop}$$

assigns to a graph with $(n+1)$-ends Γ a graph Γ' obtained by forgetting the end labeled $(n+1)$ and *stabilizing* the result if needed. This means that if a 2-valent vertex v is formed when forgetting $(n+1)$, it should be demoted to an ordinary point. Then:

(1) If v separated two edges of length d_1 and d_2, we now have only one edge of length $d_1 + d_2$;
(2) If v was adjacent to an edge and an end, then only the end is left.

See Figure 2.4 for an illustration.

We define an embedding of $M_{0,n}^{trop}$ into $\mathbb{R}^{\binom{n}{2}}$ as follows:

$$\begin{array}{rccc} dist: & M_{0,n}^{trop} & \to & \mathbb{R}^{\binom{n}{2}} \\ & [\Gamma] & \mapsto & (d(i,j))_{(i,j)} \end{array},$$

where $d(i,j)$ denotes the distance between the i-th and the j-th end of the tropical curve using the edge metric.We will check that this is a piecewise linear embedding of $M_{0,n}^{trop}$, and see the image cannot be made into a balanced fan (this already fails at $n=4$). So we take a quotient of $\mathbb{R}^{\binom{n}{2}}$.

Define the linear function:

$$\begin{array}{cccc} \Phi: & \mathbb{R}^n & \to & \mathbb{R}^{\binom{n}{2}} \\ & (a_1,\dots,a_n) & \mapsto & (a_i+a_j)_{(i,j)} \end{array},$$

and define the quotient space $Q = \mathbb{R}^{\binom{n}{2}}/Im(\Phi)$.

For any subset $I \subset [n]$ such that $2 \le |I| \le n-2$, consider the tropical curve Γ_I consisting of two vertices joined by one edge of length one, the marks in I attached to one vertex, and the marks in I^c attached to the other. We define

$$\mathbf{v}_I := dist(\Gamma_I) \subset \mathbb{R}^{\binom{n}{2}}. \tag{30}$$

*We sometimes will want to think of $\mathbf{v}_I$ as living in the quotient space Q; we will let the context make that distinction. To avoid too much notation, we will also call **dist** the composition of the map **dist** with the projection function to Q. The map $\boldsymbol{dist}: M_{0,n}^{trop} \to Q$ is still an injective function. Here we are going to take it for granted.*

EXAMPLE 2.1. Consider the function $dist: M_{0,4}^{trop} \to \mathbb{R}^6$. We have

$$\mathbf{v}_{\{12\}} = [0,1,1,1,1,0]$$
$$\mathbf{v}_{\{13\}} = [1,0,1,1,0,1]$$
$$\mathbf{v}_{\{14\}} = [1,1,0,0,1,1].$$

The function $dist$ is injective, and linear on each cone of $M_{0,4}^{trop}$. There are no (non-negative) weights that one may give the cones of $dist(M_{0,4}^{trop})$ which will make the image a balanced fan in $\mathbb{R}^6$. However notice that

$$\mathbf{v}_{\{12\}} + \mathbf{v}_{\{13\}} + \mathbf{v}_{\{14\}} = [2,2,2,2,2,2] = \Phi(1,1,1,1)$$

lies in the image of Φ, hence $M_{0,4}^{trop}$ is a balanced fan in Q when all cones are given weight one.

Consider a tropical curve $[\Gamma] \in M_{0,n}^{trop}$ of a given topological type T (you may assume T is chosen so that Γ has no edges of length 0), and assume that $dist([\Gamma])$ has integral coordinates. In Exercise 7.9, you will see this implies that all edges of Γ have integral length. It follows that if one chooses the vectors $\mathbf{v}_{I_e}$ (for e varying among all the edges of Γ) as the markings on the rays of the cone $\tau = dist(C_T)$, then the weight of τ is equal to 1.

THEOREM 2.1. *The image of $dist(M_{0,n}^{trop})$, with all cones taken with weight one, is a balanced, marked fan in Q.*

PROOF. By Lemma 1.1, we must show, that for every codimension one cone $\tau \in dist(M_{0,n}^{trop})$, for every top dimensional cone $\sigma > \tau$, the sum of the $\mathbf{v}_{\sigma\smallsetminus\tau}$ lies in the span of τ inside Q.

A codimension one cone τ parameterizes curves with exactly one four-valent vertex, and all other vertices trivalent. Call A, B, C, D the four components of the graph attached to the four-valent vertex. Note that A, B, C, D are either ends or trivalent trees. There are three top dimensional cones σ

that contain τ as a face, consisting of the three possible ways of separating the ends of the four-valent vertex into two pairs.

We want to study separately two cases. The first case is when A, B, C, D are all trivalent trees. In this case we have that the (ij) coordinate of the vector

$$[\mathbf{v}_{A\cup B} + \mathbf{v}_{A\cup C} + \mathbf{v}_{A\cup D}]_{(ij)} = \begin{cases} 2 & \text{if } i,j \text{ belong to different components} \\ 0 & \text{if if } i,j \text{ belong to the same component.} \end{cases}$$

Consider a curve Γ whose image lies in τ, such that the lengths of the four edges adjacent to the four-valent vertex are one, and all other egdes have length 0. Then

$$[dist(\Gamma)]_{(ij)} = \begin{cases} 2 & \text{if } i,j \text{ belong to different components} \\ 0 & \text{if if } i,j \text{ belong to the same component.} \end{cases}$$

It follows that $\mathbf{v}_{A\cup B} + \mathbf{v}_{A\cup C} + \mathbf{v}_{A\cup D} = dist(\Gamma)$, so in particular it belongs to $Span(\tau)$. We leave it as an exercise to modify the proof to address the case when some of the components are just single ends. $\square$

From now on by $M_{0,n}^{trop}$ we will mean the balanced fan in Q thus obtained.

LECTURE 3

Moduli Spaces of Tropical Stable Maps

In this lecture we introduce the moduli spaces of tropical stable maps: abstract tropical curves toghether with a function that maps them into the plane in an appropriate" way. We will learn that these moduli spaces are quite similar to our old friends $M_{0,n}^{trop}$'s. In particular, they also are balanced fans. These moduli spaces admit evaluation functions, that allow us to identify where in the plane certain points on the tropical curves are mapped. By combining several evaluation functions, we can construct a map of balanced fans whose degree computes the number of tropical curves of a certain degree that pass through a certain number of fixed points in the plane.

Definition 3.1. A **tropical, rational, n-marked stable map to the plane** consists of a tuple (Γ, φ), where:

- $\Gamma \in M_{0,n+m}^{trop}$ is an abstract, stable, tropical, rational curve with $n+m$ ends, with $n \geq 0$ and $m \geq 2$;
- $\varphi : \Gamma \to \mathbb{R}^2$ is a continuous function which restricts to an integral affine linear function on each edge or end of Γ (i.e. if t is the length coordinate on the edge, the restriction of φ is of the form $\varphi(t) = \mathbf{v}t + \mathbf{a}$, with $\mathbf{v} \in \mathbb{Z}^2$).

We call $\mathbf{v}$ as above the **direction vector** of the edge or end. Note that in fact the sign of the direction vector depends on the choice of an orientation for the edge. We also have the following requirements:

balancing: for every vertex v of Γ, the sum of the direction vectors of the adjacent edges/ends, oriented outgoing from the vertex, is $\mathbf{0}$:

$$\sum_{e \ni v} \mathbf{v}_e = \mathbf{0}. \tag{31}$$

marks: the direction vector for each of the first n-ends equals $\mathbf{0}$.

ends: the direction vectors for the last m-ends are different from $\mathbf{0}$.

We see an example of a tropical stable maps in Figure 3.1. For simplicity of notation, we often draw a tropical stable maps just by drawing the image of the map, and we omit the labeling of the m ends that are not contracted.

Remark 3.1. Let (Γ, φ) be a tropical stable map. We may give weights to the edges of $\varphi(\Gamma)$ as follows: if $\mathbf{v_e} = (x_e, y_e)$ is the direction vector associated to an edge e, we define the weight $w_e = gcd(x_e, y_e)$. With this convention, the image curve $\varphi(\Gamma)$ satisfies the balancing condition for embedded tropical curves as stated in Diane's mini-course. We recall it here for convenience.

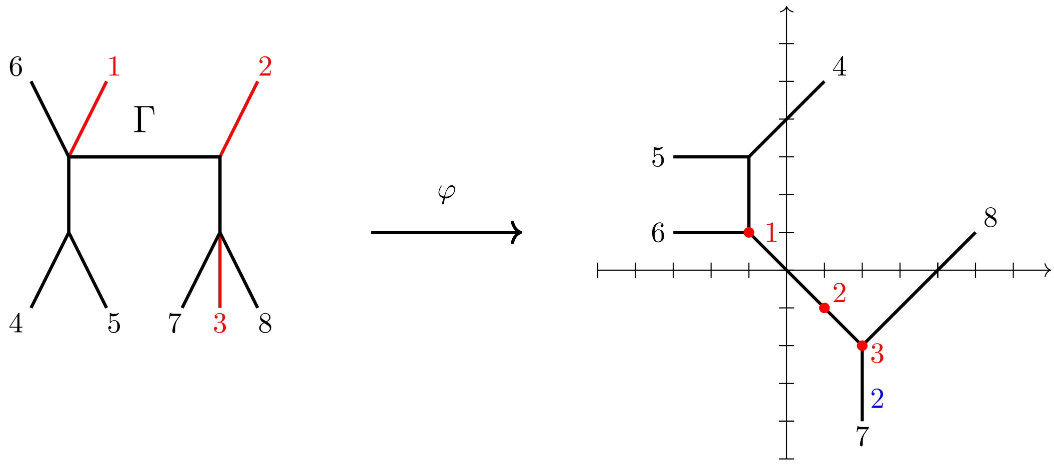


FIGURE 3.1. An example of a tropical stable map. The red marked ends are contracted to points. The blue 2, which can be thought as a weight on an end, reminds us that the direction vector for that end is twice the primitive vector in that direction.

Let Γ be a tropical plane curve. For any oriented edge e of Γ, denote by $\mathbf{p}_e$ the primitive vector of e and w_e the weight of e. For any vertex v of Γ, let $e_1, \dots e_k$ be the edges of Γ adjacent to v, oriented away from the vertex. Then we say Γ is **balanced at** v if

$$\sum_{i=1}^{k} w_{e_i} \mathbf{p}_{e_i} = 0. \tag{32}$$

DEFINITION 3.2. The ordered list of non-zero direction vectors of the ends of Γ is called the **degree** of the map φ and denoted by Δ.

DEFINITION 3.3. We denote by $M_{0,n}^{trop}(\mathbb{R}^2, \Delta)$ the parameter space of tropical rational, n-marked stable maps to the plane of degree Δ. In particular, denoting by $\mathbf{e_1}$ and $\mathbf{e_2}$ the standard basis of $\mathbb{R}^2$, if Δ consists of d copies of the vector $-\mathbf{e_1}$, d copies of the vector $-\mathbf{e_2}$ and d copies of the vector $\mathbf{e_1} + \mathbf{e_2}$, then we say the maps have degree d, and the target is tropical $\mathbb{P}^2$, and denote the space $M_{0,n}^{trop}(\mathbb{P}^2, d)$.

DEFINITION 3.4. For $i = 1, \dots, n$ we define the i-th **evaluation morphism**

$$ev_i : M_{0,n}^{trop}(\mathbb{R}^2, \Delta) \to \mathbb{R}^2 \tag{33}$$

by

$$ev_i(\Gamma, \varphi) := \varphi(i), \tag{34}$$

where i denotes the i-th marked end (which is contracted to a point by φ).

THEOREM 3.1. *Denote by $m = |\Delta|$. Then the function*

$$s \times ev_1 : M_{0,n}^{trop}(\mathbb{R}^2, \Delta) \to M_{0,n+m}^{trop} \times \mathbb{R}^2 \tag{35}$$

defined by

$$(\Gamma, \varphi) \mapsto (\Gamma, \varphi(1)) \tag{36}$$

is a bijection.

This theorem allows us to identify the moduli space of tropical stable maps with the product of a moduli space of tropical curves times a vector space. In particular, since we showed that $M_{0,n}^{trop}$ is a balanced fan, so is the moduli space of stable maps!

PROOF. The theorem is proven by explicitly constructing an inverse function. Given a curve $\Gamma \in M_{0,n+m}^{trop}$, and a point $P \in \mathbb{R}^2$, we wish to construct a tropical stable map $\varphi : \Gamma \to \mathbb{R}^2$ in $M_{0,n}^{trop}(\mathbb{R}^2, \Delta)$. Note that the direction vectors for the last m ends are specified by Δ. The balancing condition at all vertices uniquely determines all other direction vectors. Therefore φ is determined up to a global translation in $\mathbb{R}^2$. Imposing that the vertex adjacent to the first mark maps to P fixes uniquely such translation, and yields a unique map φ. □

Evaluation functions are linear functions when restricted to cones τ of $M_{0,n}^{trop}(\mathbb{R}^2\Delta) \cong M_{0,n+m}^{trop} \times \mathbb{R}^2$. For this purpose, recall from (62) that we can use the vectors $\mathbf{v}_{I_e}$ as a basis for $Span(\tau) \subset Q$ and the length of edges l_e as dual coordinates. Then, for any $(\Gamma, (x, y)) \in \tau \times \mathbb{R}^2$, let $P_{1\to i}$ be the unique oriented path from the vertex adjacent to the first mark to the vertex adjacent to the i-th mark; then we have

$$ev_i(\Gamma, (x, y)) = \begin{bmatrix} x \\ y \end{bmatrix} + \sum_{e \in P_{1\to i}} l_e \mathbf{v_e}. \tag{37}$$

In fact a stronger statement holds. We state it here and leave the proof as an exercise (see Exercise 8.10).

LEMMA 3.1. *The maps $ev_i : M_{0,n}^{trop}(\mathbb{R}^2, \Delta) \to \mathbb{R}^2$ are restriction of integral linear functions $L_i : Q \times \mathbb{R}^2 \to \mathbb{R}^2$.*

Consider the special case in which $n = |\Delta| - 1$. In this case the dimension of $M_{0,n}^{trop}(\mathbb{R}^2, \Delta)$ is equal to $2(|\Delta| - 1)$. Consider the linear function

$$Ev := ev_1 \times \ldots \times ev_n : M_{0,n}^{trop}(\mathbb{R}^2, \Delta) \to \mathbb{R}^2 \times \ldots \times \mathbb{R}^2. \tag{38}$$

The function $Ev : M_{0,n}^{trop}(\mathbb{R}^2, \Delta) \to Ev_*(M_{0,n}^{trop}(\mathbb{R}^2, \Delta))$ is a map of balanced fans, which in particular implies that Ev is surjective [1]. By Definition 1.8 we know that we have a well defined notion of degree of Ev, and Remark 1.1 tells us how to compute it!

DEFINITION 3.5. Given any degree Δ, and all notation as above,

$$N_{\Delta}^{trop} := deg(Ev), \tag{39}$$

gives the (weighted) number of tropical maps of degree Δ whose image in $\mathbb{R}^2$ is a tropical curve passing through $|\Delta| - 1$ points in general position in the plane.

[1] Technically this is true so long as Ev does not contract all maximal cones of $M_{0,n}^{trop}(\mathbb{R}^2, \Delta)$. We leave this as an exercise (see Exercise 8.8).

LECTURE 4

Kontsevich/Mikhalkin's theorem

In this last lecture we deduce a recursion that counts the number of tropical rational plane curves through $3d - 1$ points in the plane.

THEOREM 4.1 (Mikhalkin). *The numbers N_d^{trop} of rational tropical plane curves of degree d through $3d - 1$ points satisfy the recursion:*

$$N_d^{trop} = \sum_{\substack{d_1 + d_2 = d \\ d_1 \geq 1, d_2 \geq 1}} \left[\binom{3d-4}{3d_1 - 2} d_1^2 d_2^2 - \binom{3d-4}{3d_1 - 1} d_1^3 d_2 \right] N_{d_1}^{trop} N_{d_2}^{trop}. \tag{40}$$

Algebraic curves satisfy an identical recursion. Further, there is one tropical line in the plane passing through two points, as well as one algebraic line passing through two points, giving immediately the following corollary.

COROLLARY 4.1.1. *Denoting by N_d the number of rational plane (complex algebraic) curves of degree d passing through $3d - 1$ points in general position, we have: For any $d \geq 1$,*

$$N_d = N_d^{trop}. \tag{41}$$

The strategy of proof is the following. We write down a map of constant degree $\Pi : M_{0,3d}^{trop}(\mathbb{P}^2, d) \to \mathbb{R}^{6d-2} \times M_{0,4}^{trop}$. Then we fix a generic point of $\mathbb{R}^{6d-2}$ and consider two points Γ_1, Γ_2 on different ends of $M_{0,4}^{trop}$: their inverse images, counted with multiplicities, must give the degree of Π, so in particular must be equal. Formula (40) then falls out from an explicit analysis of the maps in the two inverse images, and their multiplicities.

STEP I: CONSTRUCTION OF Π.

Let $d \geq 2$ and $n = 3d$, and consider the morphism:

$$\Pi : M_{0,3d}^{trop}(\mathbb{P}^2, d) \to \mathbb{R} \times \mathbb{R} \times \underbrace{\mathbb{R}^2 \times \ldots \times \mathbb{R}^2}_{(3d-2)\ copies} \times M_{0,4}^{trop} \tag{42}$$

defined as:

$$\Pi = ev_{1,x} \times ev_{2,y} \times ev_3 \times \ldots \times ev_{3d} \times f_4, \tag{43}$$

where:

- $ev_{1,x}$ is the function that evaluates only the first coordinate of the first mark.
- $ev_{2,y}$ is the function that evaluates only the second coordinate of the second mark.

FIGURE 4.1. The curves $\gamma_1, \gamma_2 \in M_{0,4}^{trop}$.

- for $3 \leq i \leq n$, ev_i is the ordinary evaluation function, evaluating both coordinates of the i-th mark.
- f_4 is the forgetful morphism that forgets all marks $i \geq 5$, as well as the map φ.

STEP II: THE DEGREE OF Π.

The degree of Π is computed by fixing a (general) point in the codomain and counting the inverse images with appropriate multiplicities (as in Definition 1.8).

This will be a fairly long and technical computation, so we break it up into more sub-steps.

STEP II.A: TWO POINTS ON $M_{0,4}^{trop}$.

Let $\gamma_1 \in M_{0,4}^{trop}$ be a four pointed tropical rational curve where the first and second mark are on one vertex, the third and fourth on the other vertex, and the compact edge is *very long*. Let $\gamma_2 \in M_{0,4}^{trop}$ be a four pointed tropical rational curve where the first and third mark are on one vertex, the second and fourth on the other vertex, and the compact edge is *very long*. See Figure 4.1.

In this context, *very long* has a precise technical meaning. If one chooses the lengths of the compact edges of Γ_1, Γ_2 large enough, then all maps in the inverse images $\Pi^{-1}(x_0, y_0, P_3, \ldots P_n, \gamma_i)$ must contract (at least) one edge, and we say the edge is very long when this happens (see Exercise 9.2).

STEP II.B: THE MULTIPLICITY OF $(\Gamma, \varphi) \in \Pi^{-1}(x_0, y_0, P_3, \ldots P_n, \gamma_1)$.

Let (Γ, φ) be a tropical map in the inverse image $\Pi^{-1}(\gamma_1)$. We want to compute $mult_{\Pi}(\Gamma, \varphi)$. We know that Γ must have an edge contracted by φ, and such edge separates the marks $1, 2$ from the marks $3, 4$.

We study separately two cases. First, when the contracted edge is adjacent to the marks 1 and 2.

Let $(\Gamma, \varphi) \in \Pi^{-1}(x_0, y_0, P_3, \ldots P_n, \gamma_1)$ be such that the contracted edge e is adjacent to the marks $1, 2$, as in Figure 4.2. Note that the whole tripod consisting of the edge e together with the ends labeled 1,2 must be contracted to the point (x_0, y_0). Replacing this tripod with a contracting end, we recognize that $(\tilde{\Gamma}, \tilde{\varphi})$ is a curve contributing to the count of rational curves of

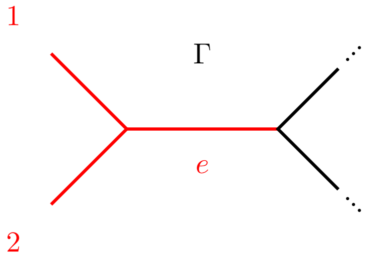


FIGURE 4.2. A part of the source curve Γ for a tropical stable map $(\Gamma, \varphi) \in \Pi^{-1}(x_0, y_0, P_3, \dots P_n, \gamma_1)$, where the edge e which is contracted by φ is adjacent to the first two marks. The dots signify that the curve Γ continues.

$$M_{(\Gamma,\varphi)}(\Pi) = \left[\begin{array}{ccc|c} & & & 0 \\ & M_{(\tilde{\Gamma},\tilde{\varphi})}(Ev) & & 0 \\ & & & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

FIGURE 4.3. The relationship between matrices computing the multipilicities of the maps Π and Ev.

degree d through $3d-1$ points. Viceversa, for any $(\tilde{\Gamma}, \tilde{\varphi})$ contributing to the count of rational curves of degree d through $3d-1$ points, by replacing the mark 0 with a tripod, one obtains a point $(\Gamma, \varphi) \in \Pi^{-1}(x_0, y_0, P_3, \dots P_n, \gamma_1)$ such that the contracted edge e is adjacent to the marks $1, 2$.

The matrix that computes $mult_\Pi(\Gamma, \varphi)$ and the matrix that computes the contribution of $(\tilde{\Gamma}, \tilde{\varphi})$ to the count N_d^{trop} have the same determinant (see Figure 4.3), hence the contribution to the weighted count of points in $\Pi^{-1}(x_0, y_0, P_3, \dots P_n, \gamma_1)$ by curves where the contracted edge is adjacent to the marks $1, 2$ equals N_d.

The next case is when the contracted edge e is not adjacent to the marks $1, 2$ in Γ, but somewhere in the middle of the curve Γ.

LEMMA 4.1. *Let $(\Gamma, \varphi) \in \Pi^{-1}(x_0, y_0, P_3, \dots P_n, \gamma_1)$ with non-zero contribution to $deg(\Pi)$ be such that the contracted edge e is not adjacent to the marks $1, 2$.*

(1) *By cutting the edge e one obtains two maps $(\Gamma_1, \varphi_1), (\Gamma_2, \varphi_2)$ of degree d_1, d_2 with $d_1 + d_2 = d$.*
(2) *The curve Γ_1 contains the marks $1, 2$ and $3d_1 - 1$ further markings.*
(3) *The curve Γ_2 contains the marks $3, 4$ and $3d_2 - 3$ further markings.*
(4) *Denote by L_1 the line $x = x_0$ and by L_2 the line $y = y_0$. Also, to avoid excessive notation, denote by Γ_i both the source of the map and the image plane tropical curve. We have:*

$$mult_{(\Gamma,\varphi)}(\Pi) = mult_{(\Gamma_1,\varphi_1)}(Ev) \cdot mult_{(\Gamma_2,\varphi_2)}(Ev) \cdot mult_{\varphi(e)}(\Gamma_1, \Gamma_2) \cdot \\ mult_{\varphi(1)}(\Gamma_1, L_1) \cdot mult_{\varphi(2)}(\Gamma_2, L_2). \tag{44}$$

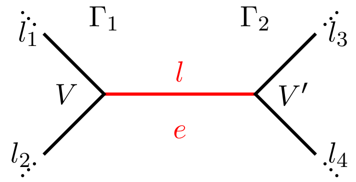


FIGURE 4.4. A part of the source curve Γ for a tropical stable map $(\Gamma, \varphi) \in \Pi^{-1}(x_0, y_0, P_3, \dots P_n, \gamma_1)$, where the contracted edge e is not adjacent to the first two marks. The edges of lengths l_1 and l_2, oriented outwards from V, are mapped via φ with direction vectors $\pm\mathbf{v}$. The edges of lengths l_3, l_4, are mapped via φ with direction vectors $\pm\mathbf{w}$.

PROOF. The first statement follows from the balancing condition: when cutting the edge e, Γ breaks into two connected components, and the sum of the direction vectors of the ends for each of the components must be zero. Therefore each component must contain the same number of down, left and diagonal ends, showing that the restriction of φ to each component is a tropical map to the tropical projective plane.

For the second and third statements, since the edge e separates the marks $1, 2$ and $3, 4$ we are just choosing to call Γ_1 the connected component containing the first two marks and Γ_2 the connected component containing the third and fourth. For the second part of these statements, notice that since the first two marked points are required to map to lines, this does not put any restriction on the map (Γ_1, φ_1). Therefore we may impose that a map of degree d_1 passes through at most $3d_1 - 1$ marks. On the other hand, the third and fourth marked points are required to map to fixed points P_3, P_4, and therefore we may impose that a map of degree d_2 passes through at most $3d_2 - 3$ further points. But since the total of marked points of Γ is $3d$, the maximum constraint must be attained by each component (since if one component attains less than the maximum, then the other is required more than its maximum).

We now come to analyse the multiplicity of the map Π. Observe (See Exercise 8.7) that the choice of using the first vertex for fixing the translation factor in the bijection $M_{0,3d}^{trop}(\mathbb{P}^2, d) \to M_{0,6d}^{trop} \times \mathbb{R}^2$ is irrelevant: choosing any other vertex does not alter the determinant of the matrix M. We choose to evaluate the vertex V adjacent to the edge e and belonging to Γ_1.

Figure 4.4 is a local picture of Γ around the contracted edge e. We rearrange rows and columns of M so as to obtain a block decomposed matrix that looks as depicted in Figure 4.5.

Here is how to interpret this matrix: we have no control of the blocks filled with $*$'s. For each marked point, the corresponding ev_i gives two rows of the matrix, to be chosen among the four depicted in the matrix based on whether the i-th mark lives in Γ_1 or Γ_2, and whether it preceeds or follows the edge e. We are assuming here that the marks 1 and 2 are both behind

	l	lengths in Γ_1	l_1	l_2	l_3	l_4	lengths in Γ_2	$\varphi(V)$
$ev_{1,x}$	0	$*$	v_x	0	0	0	0	1 0
$ev_{2,y}$	0	$*$	v_y	0	0	0	0	0 1
evaluations of coordinates of points behind l_1	0	$*$	$\mathbf{v}$	0	0	0	0	$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$
evaluations of coordinates of points behind l_2	0	$*$	0	$-\mathbf{v}$	0	0	0	$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$
evaluations of coordinates of points behind l_3	0	0	0	0	$\mathbf{w}$	0	$*$	$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$
evaluations of coordinates of points behind l_4	0	0	0	0	0	$-\mathbf{w}$	$*$	$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$
f_4	1	$*$	$*$	$*$	$*$	$*$	$*$	$*$

FIGURE 4.5. The matrix computing $mult_{(\Gamma,\varphi)}(\Pi)$.

l_1. If one follows l_1 and the other l_2, the matrix needs to be slightly changed but the proof will follow the same way.

This is a $6d-1 \times 6d-1$ matrix. For the purposes of calculating the determinant we can remove the first column and the last row. Now we perform the following column operations that do not change the absolute value of the determinant:

- Move the last two columns to the front;
- Multiply the first column of $\varphi(V)$ by v_x, the second column by v_y and add them to the column l_2; subtract the column l_1;
- Subtract l_4 from l_3;
- Since $\mathbf{v}, \mathbf{w}$ are linearly independent, there exists a linear combination $c_1(\mathbf{v}, \mathbf{w})$ that equals the first standard basis vector $\mathbf{e_1}$, and another linear combination $c_2(\mathbf{v}, \mathbf{w})$ that equals $\mathbf{e_2}$. Subtract $c_1(\tilde{l}_2, l_4 - l_3)$ from the first column and $c_2(\tilde{l}_2, l_4 - l_3)$ from the second.

The resulting matrix is depicted in Figure 4.6.

	$\varphi(V) - C(\tilde{l}_2, l_4 - l_3))$	lengths in Γ_1	l_1	$\tilde{l}_2 = l_2 - l_1 + \varphi(V)\mathbf{v}$	$l_3 - l_4$	l_4	lengths in Γ_2
$ev_{1,x}$	1 0	$*$	v_x	0	0	0	0
$ev_{2,y}$	0 1	$*$	v_y	0	0	0	0
evaluations of coordinates of points behind l_1	$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$	$*$	$\mathbf{v}$	0	0	0	0
evaluations of coordinates of points behind l_2	$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$	$*$	0	0	0	0	0
evaluations of coordinates of points behind l_3	0	0	0	$\mathbf{v}$	$\mathbf{w}$	0	$*$
evaluations of coordinates of points behind l_4	0	0	0	$\mathbf{v}$	$\mathbf{w}$	$-\mathbf{w}$	$*$

FIGURE 4.6. After performing column operations, the matrix computing $mult_{(\Gamma,\varphi)}(\Pi)$ decomposes.

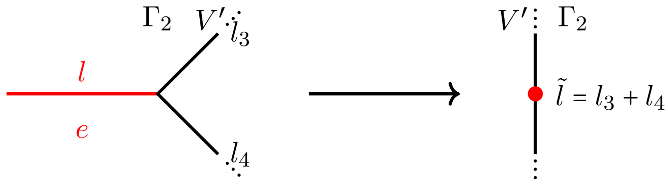


FIGURE 4.7. The tropical curve Γ_2 does not have the edge e. Therefore the two adjacent edges are replaced by a unique edge of length $\tilde{l} = l_3 + l_4$.

We now have a block decomposition of the matrix. Let us first focus on the lower block M_{SE}, and observe that it is very similar to the matrix A_2 computing the multiplicity of Ev for the curve Γ_2.

Referring to Figure 4.7 for notation, the matrix A_2 is:

$A_2 =$

	$\varphi(V')$	$\tilde{l}$	lengths in Γ_2
evaluations of coordinates of points before V'	$\begin{matrix}1 & 0\\0 & 1\end{matrix}$	0	$*$
evaluations of coordinates of points after V'	$\begin{matrix}1 & 0\\0 & 1\end{matrix}$	$-\mathbf{w}$	$*$

Let $M_{\mathbf{v},\mathbf{w}}$ be a block matrix consisting of a 2×2 upper block containing the coordinates of the vectors $\mathbf{v},\mathbf{w}$ and a lower identity block. Then we have:

$$M_{SE} = A_2 \cdot M_{\mathbf{v},\mathbf{w}} \tag{45}$$

and therefore

$$det(M_{SE}) = det(A_2)det(M_{\mathbf{v},\mathbf{w}}) = mult_{(\Gamma_2,\varphi_2)}(Ev)\cdot mult_{\varphi(e)}(\Gamma_1,\Gamma_2). \tag{46}$$

We now turn to the north-west block of M, which we denote M_{NW}. This is a $6d_1\times 6d_1$ matrix and it is quite similar to the matrix A_1 computing the multiplicity of Ev for Γ_1: the problem is that A_1 is a square matrix of size $6d_1 - 2$. The issue arises with the fact that the marks 1 and 2, which are mapping to L_1 and L_2, are subdividing two edges of Γ, and A_1 does not see this subdivision, as shown in Figure 4.8 . Denote by $\mathbf{s}$ the direction vector of the edge of Γ_1 containing the first mark, and denote ℓ_1^+,ℓ_1^- the lengths of the two adjacent edges. Then we have:

$M_{NW} =$

	$\varphi(V)$	ℓ_1^-	ℓ_1^+	other lengths in Γ_1	l_1
$ev_{1,x}$	$1\quad 0$	s_x	0	$*$	v_x
$ev_{2,y}$	$0\quad 1$	$\star$	$\star$	$*$	v_y
evaluations of coordinates of points before 1	$\begin{matrix}1 & 0\\0 & 1\end{matrix}$	0	0	$*$	$\mathbf{v}$
evaluations of coordinates of points after 1	$\begin{matrix}1 & 0\\0 & 1\end{matrix}$	$\mathbf{s}$	$\mathbf{s}$	$*$	$\mathbf{v}$

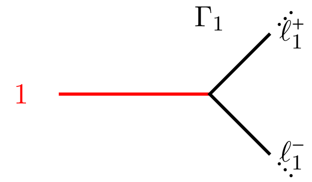


FIGURE 4.8. The portion of tropical curve Γ_1 near the first mark.

Subtracting ℓ_1^+ from ℓ_1^- we get:

	$\varphi(V)$	$\ell_1^- - \ell_1^+$	ℓ_1^+	other lengths in Γ_1	l_1
$ev_{1,x}$	$1 \quad 0$	s_x	0	$*$	v_x
$ev_{2,y}$	$0 \quad 1$	0	$\star$	$*$	v_y
evaluations of coordinates of points before 1	$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$	0	0	$*$	$\mathbf{v}$
evaluations of coordinates of points after 1	$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$	0	$\mathbf{s}$	$*$	$\mathbf{v}$

,

which tells us that the absolute value of the determinant of M_{NW} equals $|s_x|$ times the determinant of the matrix obtained by removing the row corresponding to $ev_{1,x}$ and the column $\ell_1^- - \ell_1^+$. We note that $|s_x| = mult_{\varphi(1)}(\Gamma_1, L_1)$. A similar argument applies for the the row $ev_{2,y}$, and the remaining matrix is precisely A_1. Therefore we have:

$$det(M_{NW}) = mult_{(\Gamma_2,\varphi_2)}(Ev) \cdot mult_{\varphi(1)}(\Gamma_1, L_1) \cdot mult_{\varphi(2)}(\Gamma_1, L_2). \quad (47)$$

The lemma is now proven by combining (46) and (47). □

STEP II.C: ADDING ALL THE MULTIPLICITIES IN $\Pi^{-1}(x_0, y_0, P_3, \ldots P_n, \gamma_1)$

Choose $3d_1 - 1$ points among $P_3, \ldots, P_n$ and call them $Q_1, \ldots Q_{3d_1-1}$ (call $R_1, \ldots R_{3d_2-3}$ the complementary points); consider the two following sets:

- $X_1 = \{(\Gamma_1, \varphi_1) \in M^{trop}_{0,3d_1+1}(\mathbb{P}^2, d_1) | \varphi_1(i) = Q_i, \varphi_1(3d_1) \in L_1, \varphi(3d_1 + 1) \in L_2\}$;
- $X_2 = \{(\Gamma_2, \varphi_2) \in M^{trop}_{0,3d_2-1}(\mathbb{P}^2, d_1) | \varphi_1(i) = R_i, \varphi_1(3d_2 - 2) = P_1, \varphi(3d_2 - 1) = P_2\}$.

We compute the weighted sums:

$$\sum_{X_2} mult_{(\Gamma_2,\varphi_2)}(Ev) = N_{d_2}, \quad (48)$$

and

$$\sum_{X_1} mult_{(\Gamma_1,\varphi_1)}(Ev) \cdot mult_{\varphi_1(3d_1)}(\Gamma_1, L_1) \cdot mult_{\varphi_1(3d_1+1)}(\Gamma_1, L_2) = d_1^2 N_{d_1}, \quad (49)$$

where the factor d_1^2 comes from Bézout's theorem. Note that we can construct a correspondence:

$$\iota : X_1 \times X_2 \to \Pi^{-1}(x_0, y_0, P_3, \dots P_n, \gamma_1) \tag{50}$$

by associating to a pair $(\Gamma_1, \varphi_1), (\Gamma_2, \varphi_2)$ all the curves in $\Pi^{-1}(x_0, y_0, P_3, \dots P_n, \gamma_1)$ obtained by inserting a contracting edge of appropriate length at any of the points x of intersection of Γ_1 and Γ_2. and that the contribution to the degree of Π by elements in the image of ι is:

$$\sum_{Im(\iota)} mult_{(\Gamma_1,\varphi_1)}(Ev) \cdot mult_{(\Gamma_2,\varphi_2)}(Ev) \cdot mult_{\varphi_1(3d_1)}(\Gamma_1, L_1) \cdot$$
$$mult_{\varphi_1(3d_1+1)}(\Gamma_1, L_2) \cdot mult_x(\Gamma_1, \Gamma_2) = d_1^3 d_2 N_{d_1} N_{d_2}, \tag{51}$$

We finally note that there are $\binom{3d-4}{3d_1-1}$ ways to choose $3d_1 - 1$ points among $P_3, \dots, P_n$. The images of the correspondences ι for any two of these choices are disjoint, and the union of such images exausts all elements in $\Pi^{-1}(x_0, y_0, P_3, \dots P_n, \gamma_1)$ such that, after cutting the contracted edge, one is left with a pair of maps of degrees d_1 and d_2.

Adding over all pairs of degrees $d_1 + d_2 = d$, and remembering the contribution from when the contracted edge was adjacent to both marks $1, 2$, we obtain

$$deg(\Pi) = N_d + \sum_{\substack{d_1 + d_2 = d \\ d_1 \geq 1, d_2 \geq 1}} \left[\binom{3d-4}{3d_1-1} d_1^3 d_2 \right] N_{d_1}^{trop} N_{d_2}^{trop}. \tag{52}$$

Next one computes $deg(\Pi)$ using $\Pi^{-1}(x_0, y_0, P_3, \dots P_n, \gamma_2)$ (see Exercise 9.3). to obtain

$$deg(\Pi) = \sum_{\substack{d_1 + d_2 = d \\ d_1 \geq 1, d_2 \geq 1}} \left[\binom{3d-4}{3d_1-2} d_1^2 d_2^2 \right] N_{d_1}^{trop} N_{d_2}^{trop}. \tag{53}$$

Setting (52) and (53) equal to each other, one concludes the proof of Theorem 4.1.

Part 2

Exercises

EXERCISES 5

Preliminary Exercises

Recall some basic definitions from Diane's course.

Definition 5.1. The **tropical semifield** $\mathbb{T} = (\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$ consists of the real numbers union the symbol $-\infty$, with the two operations:

tropical sum:

$$a \oplus b := max(a, b),$$

where we understand that $-\infty$ is considered to be smaller than any real number.

tropical multiplication:

$$a \odot b := a + b.$$

Definition 5.2. A **tropical polynomial** $p(x_1, \dots, x_n)$ is a finite sum of tropical monomials:

$$p(x_1, \dots, x_n) = \bigoplus_{I=(i_1,\dots,i_n)} \alpha_I \odot x_1^{\odot i_1} \odot \dots \odot x_n^{\odot i_n}, \tag{54}$$

where all the $i_j \in \mathbb{Z}^{\geq 0}$, and $x^{\odot i} = \underbrace{x \odot \dots \odot x}_{i\text{–times}}$.

Exercise 5.1. Let $p(x) = a_0 \oplus (a_1 \odot x) \oplus \dots \oplus (a_d \odot x^{\odot d})$ be a univariate tropical polynomial of degree d. When interpreted in terms of ordinary arithmetics, it determines a piecewise (affine) linear function $f_p : \mathbb{R} \to \mathbb{R}$. Prove that the graph of f_p can have at most d corners. What conditions on the coefficients a_i must hold in order for the graph of f_p to have exactly d corners? In this case, call $r_1, \dots r_d$ the x-coordinate of the corners. Prove that p factors in linear factors:

$$p(x) = a_0 \odot (x \oplus r_1) \odot \dots \odot (x \oplus r_d). \tag{55}$$

We say that the x-coordinates of the corners are the **roots** of the tropical polynomial p.

Exercise 5.2. In analysis we define the exponential function as:

$$e^x := \sum_{n=0}^{\infty} \frac{x^n}{n!}. \tag{56}$$

Let us define the **tropical exponential function** $e^{\odot x}$ by replacing all the operations by their tropical counterparts (be careful: what do division, and factorials correspond to tropically?). Describe the graph of $e^{\odot x}$ and its roots.

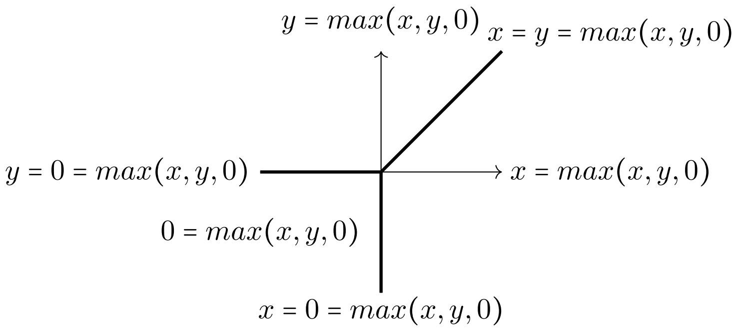


FIGURE 5.1. The tropical line determined by the tropical poynomial $p(x, y) = x \oplus y \oplus 0$. The domain plane is subdivided in three parts, according to which of the functions x, y and 0 attains the maximum value. The tropical line $V(p)$ is the tripod separating these regions.

EXERCISE 5.3. Recall that a **tropical line** associated to the tropical polynomial $p(x, y) = a \odot x \oplus b \odot y \oplus c$ may be defined in two equivalent ways:

i: the locus in the real plane where the maximum of the linear functions $a + x, b + y, c$ is attained at least twice;

ii: (the closure of) the negative valuation of the coordinates of the solutions of the equation $ax + by + c = 0$ in the field of Puiseux series.

For example, let $p(x, y) = x \oplus y \oplus 0$. The associated function is $f_p(x, y) = max(x, y, 0)$. The corresponding **tropical line** is depicted in Figure 5.1.

(1) Draw the tropical line associated to the tropical polyomial $a \odot x \oplus b \odot y \oplus c$ for $a, b, c \in \mathbb{T}$. Analyse also the case when some of the coefficients equal $-\infty$.
(2) Show that there is a unique tropical line through two general points in the plane. What does "general" mean here?
(3) Show that two general lines intersect in exactly one point. What does "general" mean here?

EXERCISE 5.4. We say that two tropical plane curves **intersect transversely** if they intersect in a finite number of points which are not vertices for either of the curves, as shown in Figure 5.2.

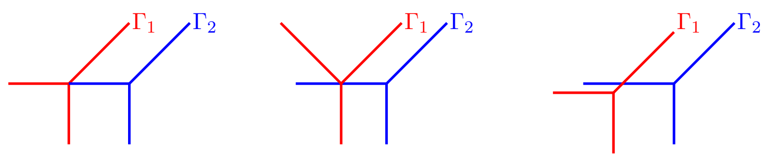


FIGURE 5.2. The leftmost tropical curves do not intersect transversely because their intersection consists of infinitely many points. The central pair also do not intersect transversely, since the intersection point is a vertex of Γ_1. The right hand side tropical curves intersect transversely.

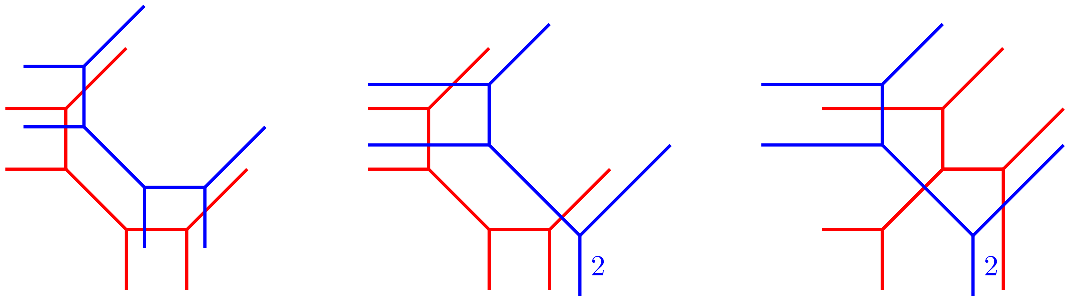


FIGURE 5.3. Compute the multiplicities of intersection of the following pairs of curves.

Let Γ_1, Γ_2 be two tropical curves, which intersect transversely at the point P. Denote by e_1 the edge of Γ_1 containing P, $\mathbf{p_1}$ the primitive vector in the direction of e_1 and w_1 the weight of e_1, and similarly for the second curve. We define the **multiplicity of intersection** of Γ_1 and Γ_2 at P to be

$$mult_P(\Gamma_1, \Gamma_2) := \left| det\left(\begin{bmatrix} x(w_1\mathbf{p_1}) & x(w_2\mathbf{p_2}) \\ y(w_1\mathbf{p_1}) & y(w_2\mathbf{p_2}) \end{bmatrix} \right) \right|. \tag{57}$$

(1) Compute $\Gamma_1 \cdot \Gamma_2$ for the pairs of curves in Figure 5.3.
(2) Prove this alternative interpretation of the intersection multiplicity of two tropical curves at a point. Given two integral vectors $\mathbf{v_1}, \mathbf{v_2} \in \mathbb{R}^2$, prove that

$$\left| det\left(\begin{bmatrix} x(\mathbf{v_1}) & x(\mathbf{v_2}) \\ y(\mathbf{v_1}) & y(\mathbf{v_2}) \end{bmatrix} \right) \right| = \left| \frac{\mathbb{Z}^2}{\mathbb{Z}\mathbf{v_1} + \mathbb{Z}\mathbf{v_2}} \right|. \tag{58}$$

This number is called the **lattice index** of the lattice generated by the vectors $\mathbf{v_1}, \mathbf{v_2}$ inside the lattice $\mathbb{Z}^2$.

EXERCISE 5.5 (Bézout's Theorem). Let Γ_1 be a tropical plane curve of degree d_1 and Γ_2 a tropical plane curve of degree d_2; assume that Γ_1 and Γ_2 intersect transversely. Then:

$$\Gamma_1 \cdot \Gamma_2 := \sum_{P \in \Gamma_1 \cap \Gamma_2} mult_P(\Gamma_1, \Gamma_2) = d_1 d_2. \tag{59}$$

(1) In Figure 5.4, v is a balanced vertex of a tropical curve Γ: the three edges have weights w_i and primitive vectors $\mathbf{p_i}$ such that $\sum w_i \mathbf{p_i} = 0$. The lines ℓ_1 and ℓ_2 are parallel. Denote P_1, P_2, Q the intersection points. Prove that

$$mult_{P_1}(\Gamma, \ell_1) + mult_{P_2}(\Gamma, \ell_1) = mult_Q(\Gamma, \ell_2). \tag{60}$$

(2) Use this idea to prove Bézout's theorem.

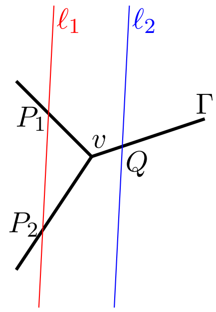


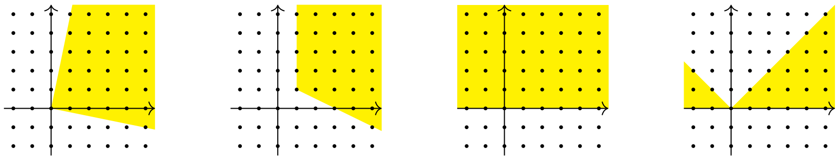
FIGURE 5.4. Local picture around the vertex of a tropical curve.

EXERCISES 6

Cone Complexes and their Functions

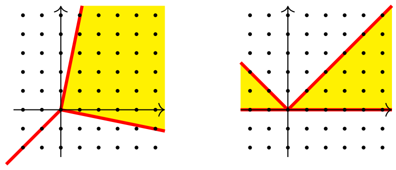
Exercise 6.1. We start with some question to solidify the intuition for the basic definitions for cones and fans.

(1) Which of the following pictures represent a rational polyhedral cone?



(2) We did not give precise definitions of the notions of **dimension** of a cone, and **face** of a cone. Given the intuitive discussions we have had about them, try and formulate precise definitions for these concepts.

(3) Which of the following pictures represent a rational polyhedral fan? Which ones are pure dimensional?



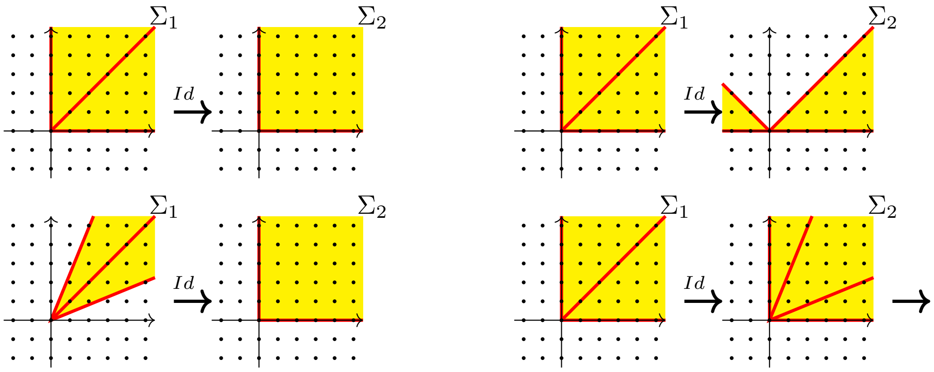
(4) Given a rational polyhedral fan Σ, we define the **support** of Σ, denoted $|\Sigma|$, to be the set of points in $\mathbb{R}^N$ that belong to some cone of Σ. Decide which of the following statements are true:
   (a) The support of Σ is a linear subspace of $\mathbb{R}^n$.
   (b) The support of Σ is a convex subset of $\mathbb{R}^n$.
   (c) If $\mathbf{x} \in |\Sigma|$ and λ is a non-negative number, then $\lambda\mathbf{x} \in |\Sigma|$.
   (d) If $|\Sigma_1| = |\Sigma_2|$, then $\Sigma_1 = \Sigma_2$.

Exercise 6.2. What does the balancing condition from Definition 1.3 state if Σ is a one-dimensional fan? What are the normal vectors to a codimension one face of Σ in this case?

Exercise 6.3. Consider a two dimensional fan $\Sigma \subseteq \mathbb{R}^3$ and assume a portion of it looks like Figure 1.1, with $\mathbf{v}_\tau = [1,0,0], \mathbf{v_1} = [1,1,0], \mathbf{v_2} = [2,2,2]$ and $\mathbf{v_3} = [1,-3,-2]$. Compute the weights $\omega_\Sigma(\sigma_i)$ for the three top dimensional cones, compute the normal vectors $\mathbf{u}_{\tau/\sigma}$ and verify that Σ is balanced at the face τ.

Exercise 6.4. Cosider a two dimensional fan Σ consisting of three rays $\rho_1 = \langle[1,0]\rangle_+, \rho_2 = \langle[2,3]\rangle_+, \rho_3 = \langle[0,1]\rangle_+$ and two dimensional cones σ_1 spanned by ρ_1 and ρ_2 and σ_2 spanned by ρ_2 and ρ_3. Show that if the two dimensional cones are given weight one, then Σ is balanced along ρ_2. Choose markings on the rays of the fan, and check that the corresponding weight function yields a fan balanced along ρ_2. However show that you may not obtain the constant weight function equal to 1 this way. Conclude that not all weight functions making Σ balanced come from markings.

EXERCISE 6.5. For each of the pictures below, consider the identity function $Id : \mathbb{R}^2 \to \mathbb{R}^2$. Decide if Id induces a map of fans as in Definition 1.5. If it does, show how the cones of Σ_1 and Σ_2 should be subdivided in order for the map of fans to send cones to cones.



EXERCISE 6.6. Is it true that every integral hyperplane in $\mathbb{R}^{K+1}$ can be moved to a coordinate hyperplane by an element of $SL(K+1,\mathbb{Z})$? Note, the coefficients being $\mathbb{Z}$ is the key here. It is straightforward to find such a matrix if you allow yourself rational coefficients. I kind of convinced myself of it, and even thought I could see a path to a proof, but I didn't actually follow it through (and to be honest, it didn't seem like the prettiest path). See if you can either find a proof, or if you can come up with a counter-example to the statement.

EXERCISE 6.7. If Σ_1 is a pure dimensional fan of dimension k, what is the dimension of $f_*(\Sigma)$?

EXERCISE 6.8. Consider the fan $\Sigma_1 \subseteq \mathbb{R}^2$, consisting of four rays generated by $\pm\mathbf{e_1}, \pm\mathbf{e_2}$. What are the conditions on the weights on the four rays for Σ_1 to be a balanced fan? Now consider the map $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) = 2x + 3y$. Describe the fan $f_*(\Sigma_1)$ and check it is balanced.

EXERCISE 6.9. Let $\Sigma \subseteq \mathbb{R}^3$ be a pure two-dimensional fan whose maximal cones consist of the twelve coordinate orthants of $\mathbb{R}^3$. Choose a marking on this fan that makes it into a balanced fan. What are the weights of the maximal cones with your choice of marking? Consider the linear function $f : \mathbb{R}^3 \to \mathbb{R}^2$ given by $f(x,y,z) = 2y + 3z$. Describe the fan $f_*(\Sigma)$.

EXERCISE 6.10. Prove Lemma 1.2. Focus on a codimension one face τ and a top dimensional face $\sigma \succ \tau$ in $f_*(\Sigma)$, and on a face $\tilde{\sigma} \in \Sigma$ such that $f(\tilde{\sigma}) = \sigma$. You may assume that you use a transformation in $SL(Span(\sigma),\mathbb{Z})$ so that τ lives in the hyperplane where the last coordinate is zero. Then compute the contribution by $\tilde{\sigma}$ to the weight of σ in $f_*(\Sigma)$, and compute the relationship between the vector $\mathbf{u}_{\tau/\sigma}$ and $f(\mathbf{u}_{\tilde{\tau}/\tilde{\sigma}})$.

Exercise 6.11. Prove Lemma 1.3 (this is actually much simpler than the previous exercise).

EXERCISES 7

Abstract tropical curves and their moduli

EXERCISE 7.1. Review the definitions of abstract tropical curve from Lecture 2.

(1) What are the topological types of abstract, rational, stable, n-pointed tropical curves, for $n = 3, 4, 5$?
(2) What are the minimum and maximum number of compact edges that an abstract, rational, stable, n-pointed tropical curve can have?

EXERCISE 7.2. Let us get comfortable with some small dimensional moduli spaces of tropical curves.

(1) Describe the space $M_{0,3}^{trop}$.
(2) Understand that the space $M_{0,5}^{trop}$ is two-dimensional, and it is (combinatorially) represented as the cone over the Petersen graph, as shown in Figure 2.3. Label the two-dimensional cones in Figure 2.3 by the topological types of the curves that they parameterize.
(3) Understand the forgetful morphism $\pi_5 : M_{0,5}^{trop} \to M_{0,4}^{trop}$.

EXERCISE 7.3. This exercise explores combinatorial properties of general spaces $M_{0,n}^{trop}$ and their forgetful morphisms.

(1) What is the dimension of $M_{0,n}^{trop}$? Give a combinatorial description of the tropical curves parameterized by the top dimensional cones, and by the codimension one cones.
(2) Show that the tropical forgetful morphisms map cones to cones. Characterize which cones it is bijective on, and which cones it contracts to lower dimensional cones.

EXERCISE 7.4. Show that for $n \geq 4$:

$$\sum_{|I|=2, 1 \notin I} \mathbf{v}_I = \Phi(1, n-3, n-3, \ldots, n-3). \tag{61}$$

Count the number of vectors in the above sum, and notice that it is exactly one higher than the dimension of Q. It follows that the vectors $\mathbf{v}_I$, with $|I| = 2, 1 \notin I$, span Q subject to the relation (61), like the rays of the fan of projective space.

EXERCISE 7.5. Prove that the map $dist : M_{0,n}^{trop} \to Q$ is an injective function.

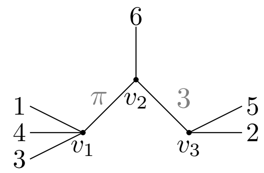
EXERCISE 7.6. Verify that the morphism π_{n+1}^{trop} functions as a universal family in the sense that:

$$\pi_{n+1}^{trop^{-1}}([\Gamma]) \cong \Gamma \subseteq M_{0,n+1}^{trop}.$$

Further, prove that π_{n+1}^{trop} is a map of fans, i.e. it is induced by a linear function.

Exercise 7.7. Show that $M_{0,n}^{trop}$ is connected through codimension one, i.e. that given any two points P, Q living in two top dimensional cones, there exists a path γ from P to Q that lives entirely in the interior of top dimensional and codimension one cones. Equivalently, this means that $M_{0,n}^{trop}$ remains connected after you remove all cones of codimension greater than one.

Exercise 7.8. Consider the tropical curve $[\Gamma] \in M_{0,6}^{trop}$ depicted in the following picture. Write down the coordinates of the vectors $dist([\Gamma]), \mathbf{v}_{134}, \mathbf{v}_{25}$ in the table. Are the three vectors linearly independent?



	(12)	(13)	(14)	(15)	(16)	(23)	(24)	(25)	(26)	(34)	(35)	(36)	(45)	(46)	(56)
$dist([\Gamma])$															
$\mathbf{v}_{\{134\}}$															
$\mathbf{v}_{\{25\}}$															

Exercise 7.9. Let Γ be an abstract, rational, tropical, n-pointed curve. Each edge e of Γ defines a two-part partition $[n] = I_e \cup I_e^c$ of the set of indices, by considering the indices that lie on either side of the edge. If we denote by $l(e)$ the length of the edge e, show that we have:

$$dist(\Gamma) = \sum_{e \in \Gamma} l_e \mathbf{v}_{I_e}. \tag{62}$$

Exercise 7.10. Given two pairs of distinct indices $i_1, t_1, i_2, t_2 \in [n]$, we define the **cross-ratio** function

$$CR_{(i_1,t_1|i_2,t_2)} : M_{0,n}^{trop} \to \mathbb{R}$$

as follows: $CR_{(i_1,t_1|i_2,t_2)}(\Gamma)$ is the signed length of $\gamma_1 \cap \gamma_2$, where γ_1 is the path from the end i_1 to the end t_1 of Γ, and similarly γ_2 is the path from the end i_2 to the end t_2 of Γ. The length is taken to be positive if the two paths have the same orientation along the intersection, and negative otherwise.

(1) Take some tropical curves, and compute a few cross-ratios, to become familiar with this definition.
(2) Prove that $CR_{(i_1,t_1|i_2,t_2)}$ is the restriction of a linear function from Q to $\mathbb{R}$.

EXERCISES 8

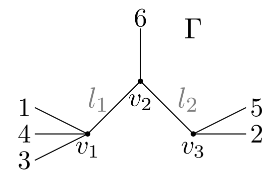
Tropical stable maps

Exercise 8.1. What is the degree of the tropical stable map depicted in Figure 3.1? What moduli space does it belong to? Using the tacks on the axes as units, compute $ev_1(\Gamma,\varphi), ev_2(\Gamma,\varphi), ev_3(\Gamma,\varphi)$ for the stable map in Figure 3.1.

Exercise 8.2. Describe explicitly $M^{trop}_{0,0}(\mathbb{P}^2, 1)$ and $M^{trop}_{0,1}(\mathbb{P}^2, 1)$.

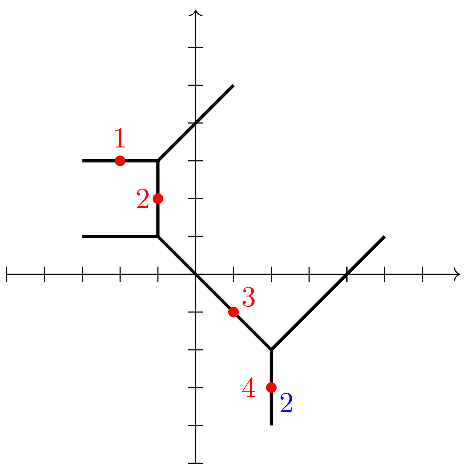
Exercise 8.3. Consider the abstract tropical curve in Figure 3.1, and the point $(0,0) \in \mathbb{R}^2$. Construct the tropical stable map of degree Δ, where $\mathbf{v}_4 = [-1,0], \mathbf{v}_5 = [-1,0]\mathbf{v}_6 = [1,1], \mathbf{v}_7 = [1,1], \mathbf{v}_8 = [0,-2]$, associated to $(\Gamma, (0,0))$ via the inverse of the function $s \times ev_1$ defined in Theorem 3.1.

Exercise 8.4. Let $\Delta = \{[-1,0],[0,-1],[1,0],[0,1]\}$. Consider the cone $\tau \times \mathbb{R}^2$ of $M^{trop}_{0,2}(\mathbb{R}^2, \Delta) \cong M^{trop}_{0,6} \times \mathbb{R}^2$ identified by the curve Γ pictured below. Recall that $\mathbf{v}_{\{134\}}, \mathbf{v}_{\{25\}}$ give a basis for the cone τ. Write down the matrices for the functions ev_1, ev_2 restricted to τ, using this basis for τ and the standard basis for $\mathbb{R}^2$.



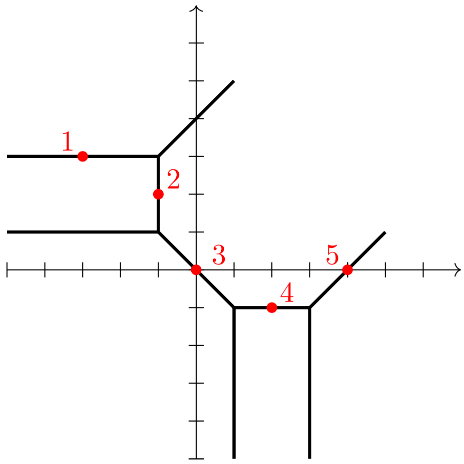
Exercise 8.5. Consider the tropical stable map $(\Gamma,\varphi) \in M^{trop}_{0,5}(\mathbb{R}^2, \Delta)$ depicted below. Compute

$$mult_{(\Gamma,\varphi)}(Ev). \tag{63}$$



EXERCISE 8.6. Consider the tropical stable map $(\Gamma, \varphi) \in M_{0,5}^{trop}(\mathbb{P}^2, 2)$ depicted below. Compute

$$mult_{(\Gamma,\varphi)}(Ev). \tag{64}$$



EXERCISE 8.7. Repeat the computation from Exercise 8.6, but pick a different bijection $M_{0,5}^{trop}(\mathbb{P}^2, 2) \to M_{0,11}^{trop} \times \mathbb{R}^2$. Instead of using ev_1 in the bijection from Theorem 3.1, use ev_3. Observe that the matrix you use to compute the multiplicity $mult_{(\Gamma,\varphi)}(Ev)$ now is obtained from the one you computed in Exercise 8.6 by column operations, and therefore the two matrices have the same determinant. Convince yourself that this is always the case, i.e. the multiplicity of Ev at a point is independent of which evaluation function we use to fix the bijection $s \times ev_i : M_{0,n}^{trop}(\mathbb{R}^2, \Delta) \to M_{0,n+|\Delta|}^{trop} \times \mathbb{R}^2$.

EXERCISE 8.8. Show that the function Ev preserves the dimension of some maximal cones of $M_{0,n}^{trop}(\mathbb{R}^2, \Delta)$ (implying in particular that the degree of Ev is not zero).

EXERCISE 8.9. Compute $deg(Ev)$ for the moduli space $M_{0,2}^{trop}(\mathbb{P}^2, 1)$. What is the geometric statement you just made?

EXERCISE 8.10. Prove Lemma 3.1. In particular, using the natural coordinates z_{ij} for $\mathbb{R}^{\binom{n}{2}}$, show that

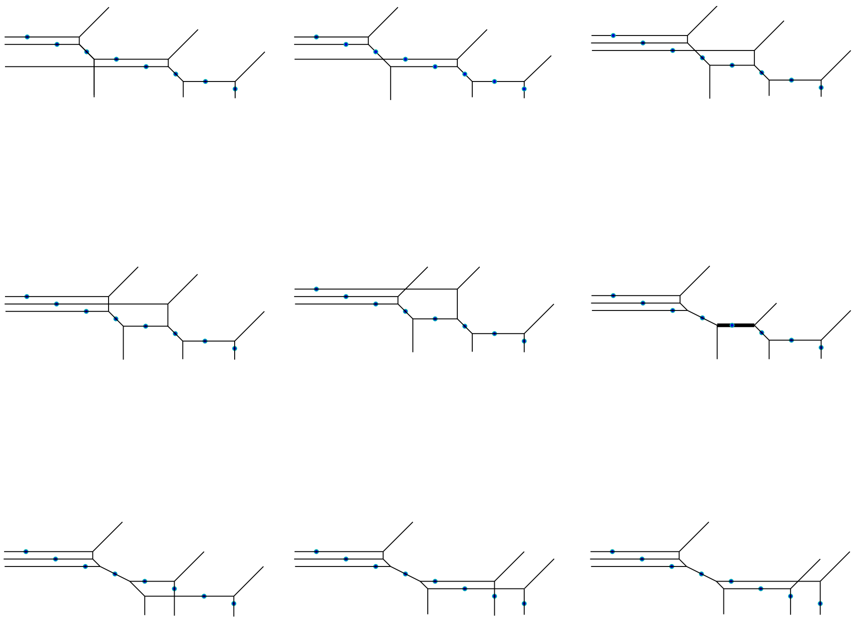
$$ev_1(\{z_{ij}\}, x, y) = \begin{bmatrix} x \\ y \end{bmatrix}, \tag{65}$$

and, for $i \neq 1$

$$ev_1(\{z_{ij}\}, x, y) = \begin{bmatrix} x \\ y \end{bmatrix} + \frac{1}{2} \sum_{\substack{k=2 \\ k \neq i}}^{n+|\Delta|} (z_{1k} - z_{ik}) \mathbf{v_k}, \tag{66}$$

where $\mathbf{v_k}$ is the direction vector assigned to the end marked by k by the direction vector Δ.

EXERCISE 8.11. The following image depicts the elements of $Ev^*(P_1, \ldots, P_8)$ for the moduli space of rational stable maps of degree 3 to the plane and 8 points in general horizontally stretched position. For each element compute its multiplicity, to obtain that $deg(Ev) = 12$.



EXERCISES 9

Mikhalkin's recursion

EXERCISE 9.1. Answer the following questions to become familiar with the definition of the auxiliary function Π.

(1) Compute $\Pi(\Gamma, f)$ for the tropical stable map in Exercise 8.6.
(2) Show that the dimension of both domain and codomain of Π is $6d-1$.
(3) Prove that the degree of Π is constant.
(4) Fix $\gamma \in M_{0,4}^{trop}$, $x_0, y_0 \in \mathbb{R}$, points $P_3, \ldots, P_n \in \mathbb{R}^2$ and consider the inverse image

$$\Pi^{-1}(\{x_0\} \times \{y_0\} \times \{P_3\} \times \ldots \times \{P_n\} \times \gamma). \tag{67}$$

Give a geometric description of the points in this inverse image.

EXERCISE 9.2. Fix values for $x_0, y_0 \in \mathbb{R}$, and points $P_3, \ldots, P_n \in \mathbb{R}^2$. Argue that there exist a uniform bound for the length of the compact egde of any curve $F(\Gamma, \varphi), \Gamma \in \Pi^{-1}(\{x_0\} \times \{y_0\} \times \{P_3\} \times \ldots \times \{P_n\} \times \gamma)$ **unless** the map $\varphi : \Gamma \to \mathbb{R}^2$ contracts some edge.

EXERCISE 9.3. Compute $deg(\Pi)$ using $\Pi^{-1}(x_0, y_0, P_3, \ldots P_n, \gamma_2)$, and obtain a proof of Theorem 4.1 by setting the two computations of $deg(\Pi)$ equal to each other. Pat yourself on the back and highfive your neighbor for making it through this intense week of math! Hopefully you learned a lot and had fun in the process!